§ 6.1 LAPLACE's EQUATION

If the diffusion or wave process is stationary, i.e., $u_t \equiv 0$, $u_{tt} \equiv 0$ both equations can be reduced to the Laplace's equation

$$u_t - k\Delta u = 0 \qquad \longrightarrow \qquad \Delta u = 0, \text{ or } \begin{cases} u_{xx} + u_{yy} = 0 & 2D \\ u_{xx} + u_{yy} + u_{zz} = 0 & 3D \end{cases}$$
$$u_{tt} - c^2 \Delta u = 0$$

Defn: A solution of the Laplace equation is called a harmonic function

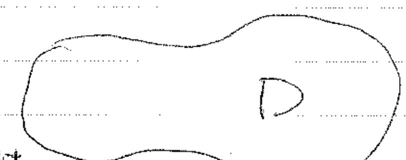
In 1D, $u_{xx} = 0 \rightarrow u(x) = Ax + B$ is a harmonic functions

Defn: $\Delta u = f$ with f a given function is called Poisson's equation

Remark: Besides stationary diffusions and waves, other examples of Laplace's and Poisson's equation include: electro statics, steady fluid flow, analytic functions, and browian motion.

$$\begin{cases} \Delta u = f & \text{in } D \\ u = h, \text{ or } \dfrac{\partial u}{\partial n} = h, \text{ or } \dfrac{\partial u}{\partial n} + au = h & \text{on bdy } D \end{cases}$$

* Maximum Principle:



Let D be a connected bounded open set. Let $u(\vec{x})$ be a harmonic function in D that is continuous on $\bar{D} = D \cup (\text{bdy } D)$. Then the maximum and the minimun values of u are attained on bdy D and nowhere inside (unless $u \equiv$ constant)

$\vec{x} = (x, y)$ in 2D, or $\vec{x} = (x, y, z)$ in 3D

$|\vec{x}| = \sqrt{x^2 + y^2}$ or $\sqrt{x^2 + y^2 + z^2}$

Mathematically, $\min\limits_{\text{bdy } D} u(\vec{x}) \le u(\vec{x}) \le \max\limits_{\text{bdy } D} u(\vec{x})$, $\forall \vec{x} \in D$

We will prove the right inequality in ~~the following~~ 2D

Proof: At a maximum point inside D, we have

$$u_{xx} \le 0 \quad \& \quad u_{yy} \le 0$$

$u_{xx} + u_{yy} \le 0$

At most maximum points, $u_{xx} < 0$ & $u_{yy} < 0$, so we get a contradiction to Laplace equation. However, it is also possible that $u_{xx} = 0 = u_{yy}$ at a maximum point.

Let $\epsilon > 0$, $v(\vec{x}) = u(\vec{x}) + \epsilon |\vec{x}|^2$, then

$$\Delta v = \Delta u + \epsilon \Delta (x^2+y^2) = 4\epsilon > 0$$

but $\Delta v = v_{xx} + v_{yy} \le 0$ at an interior maximum point

$\rightarrow$ $v(\vec{x})$ has no interior maximum point in D.

On the other side, $v(\vec{x})$ is a continuous function in the closure $\bar{D} = D \cup \text{bdy } D$. It must have a maximum at the boundary, say $\vec{x}_0 \in \text{bdy } D$. Then, $\forall \vec{x} \in D$

$$u(\vec{x}) \le v(\vec{x}) \le v(\vec{x}_0) = u(\vec{x}_0) + \epsilon |\vec{x}_0|^2 \le \max_{\text{bdy } D} u + \epsilon l^2$$

where l is the greatest distance from bdy D to the origin.

Since this is true for any $\epsilon > 0$, we have

$$u(\vec{x}) \le \max_{\text{bdy } D} u \qquad \forall \vec{x} \in D$$

The existence of a minimum point is similarly demonstrated by introducing $w = -u$. We then get $w(\vec{x}) \le \max_{\text{bdy } D} w$, $\forall \vec{x} \in D$, which further gives

$$\min_{\text{bdy } D} u \le u(\vec{x}) \qquad \forall \vec{x} \in D \qquad \#$$

* Uniqueness of the Dirichlet Problem

Suppose both u & v satisfy

$$\begin{cases} \Delta u = f & \text{in } D \\ u = h & \text{on bdy } D \end{cases} \qquad \begin{cases} \Delta v = f & \text{in } D \\ v = h & \text{on bdy } D \end{cases}$$

Then $w = u - v$ satisfies

$$\begin{cases} \Delta w = \Delta u - \Delta v = 0 & \text{in } D \\ w = u - v = h - h = 0 & \text{on bdy } D \end{cases}$$

From maximum principle, we have

$$0 = \min_{\text{bdy } D} w \le w(\vec{x}) \le \max_{\text{bdy}} w = 0 \qquad \forall \vec{x} \in D$$

$$\longrightarrow \quad w(\vec{x}) \equiv 0 \quad \& \quad u(\vec{x}) \equiv v(\vec{x}) \qquad \#$$

* Invariance

Property of Laplace operator: The Laplace operator is invariant under all rigid motions, which include translations & rotations

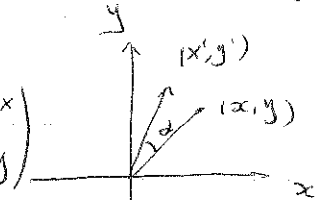
① Translation

$$\begin{cases} x' = x + a \\ y' = y + b \\ z' = z + c \end{cases} \qquad u_{xx} + u_{yy} + u_{zz} = u_{x'x'} + u_{y'y'} + u_{z'z'}$$

② Rotation

2D: $\quad x' = x\cos\alpha + y\sin\alpha$
$\quad\;\; y' = -x\sin\alpha + y\cos\alpha$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\cos\alpha + \frac{\partial}{\partial y'}(-\sin\alpha) \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial x'}\sin\alpha + \frac{\partial}{\partial y'}\cos\alpha$$

$$\frac{\partial^2}{\partial x^2} = \left[\frac{\partial}{\partial x'}\cos\alpha + \frac{\partial}{\partial y'}\sin\alpha\right]\left[\frac{\partial}{\partial x'}\cos\alpha - \frac{\partial}{\partial y'}\sin\alpha\right]$$

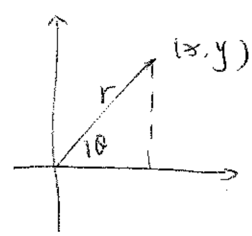
$$\frac{\partial^2}{\partial y^2} = \left[\frac{\partial}{\partial x'}\sin\alpha + \frac{\partial}{\partial y'}\cos\alpha\right]\left[\frac{\partial}{\partial x'}\sin\alpha + \frac{\partial}{\partial y'}\cos\alpha\right]$$

$$\begin{aligned} u_{xx} + u_{yy} &= u_{x'x'}(\cos^2\alpha + \sin^2\alpha) + u_{y'y'}((-\sin\alpha)^2 + \cos^2\alpha) \\ &\quad + u_{x'y'}(-\sin\alpha\cos\alpha - \sin\alpha\cos\alpha + \cos\alpha\sin\alpha + \sin\alpha\cos\alpha) \\ &= u_{x'x'} + u_{y'y'} \end{aligned}$$

This proves the invariance of the Laplace operator, ⇔ isotropy

We then introduce the polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{with Jacobian matrix}$$



$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \left[\frac{\partial(x,y)}{\partial(r,\theta)}\right]^{-1} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix}$$

then, $\displaystyle \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} + \left(-\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta} \qquad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$

$$\frac{\partial^2}{\partial x^2} = \left[\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\left[\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]$$

$$\frac{\partial^2}{\partial y^2} = \left[\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right]\left[\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right]$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

If we want to obtain special harmonic functions that are rotationally invariant, i.e., $u(x,y) = u(r,\theta) = u(r)$, then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y) = u_{rr} + \frac{1}{r} u_r = 0$$

$$\longrightarrow \quad r u_r = c_1 \quad \rightarrow \quad u = c_1 \underline{\log} r + c_2$$

3D: $\vec{x}' = B\vec{x}$ where B is an orthogonal matrix $B^T B = B B^T = I$

$$\Delta u = \sum_{i=1}^{3} u_{ii} = \sum_{i,j=1}^{3} \delta_{ij} u_{ij} \quad \text{where } \delta_{ij} \text{ is Kronecker delta}$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad i, j = 1, 2, 3$$

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = \sum_{ij} \delta_{ij} u_{ij}$$

$$\frac{\partial}{\partial x_i} = \sum_{i'} \frac{\partial}{\partial x_{i'}} \left(\frac{\partial x_{i'}}{\partial x_i}\right) = \sum_{i'} b_{ii'} \frac{\partial}{\partial x'_{i'}}$$

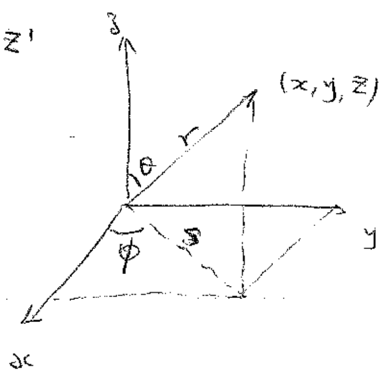
$$\frac{\partial^2}{\partial x_i \partial x_j} = \sum_{i'j'} b_{ii'} b_{jj'} \frac{\partial^2}{\partial x'_{i'} \partial x'_{j'}}$$

$$\Delta u = \sum_{ij} \sum_{i'j'} \delta_{ij} b_{ii'} b_{jj'} \frac{\partial^2}{\partial x'_{i'} \partial x'_{j'}} u = \sum_{i} \sum_{i'j'} b_{ii'} b_{ij'} \frac{\partial^2 u}{\partial x'_{i'} \partial x'_{j'}}$$

$$= \sum_{i'i'} \delta_{i'i'} \frac{\partial^2 u}{\partial x'_{i'} \partial x'_{i'}} = u_{x'x'} + u_{y'y'} + u_{z'z'}$$

$$(r, \theta, \phi) \leftarrow (s, \phi, z) \leftarrow (x, y, z)$$

Introduce the spherical coordinates (r, θ, ϕ)



$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases} \qquad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} = \sqrt{s^2+z^2} \\ s &= \sqrt{x^2+y^2} = r\sin\theta \end{aligned}$$

$$\begin{cases} x = s \cos\phi \\ y = s \sin\phi \end{cases} \qquad u_{xx} + u_{yy} = u_{ss} + \frac{1}{s} u_s + \frac{1}{s^2} u_{\phi\phi}$$

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{s} u_s + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{s^2} u_{\phi\phi}$$

Since $u_s = \frac{\partial u}{\partial s} = u_r \frac{s}{r} + u_\theta \frac{\cos\theta}{r} + u_\phi \cdot 0$

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta) u_\theta + \frac{1}{\sin^2\theta} u_{\phi\phi}\right]$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Special harmonic functions with rotational invariance

$$u(\vec{x}) = u(r,\theta,\phi) = u(r)$$

$$u_{rr} + \frac{2}{r} u_r = 0$$

$$\rightarrow \quad u = -\frac{c_1}{r} + c_2 \qquad c_1, c_2 \text{ constants}$$

Remark: $\log\sqrt{x^2+y^2}$ & $\frac{1}{\sqrt{x^2+y^2+z^2}}$ have singularities around the origin.