§ Laplace Transform

In this section we apply Lapace Transform to the time rather than the space variable.

For a function $f(t)$, Laplace transform is defined as

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

If $f(t)$ is a bounded function, then $F(s)$ is defined for $s>0$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\cos wt$	$\frac{s}{s^2+w^2}$
$\sin wt$	$\frac{w}{s^2+w^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
t^k	$\frac{k!}{s^{k+1}}$
$H(t-b)$	$\frac{1}{s} e^{-bs}$
$\delta(t-b)$	e^{-bs}
$a(4\pi t^3)^{-\frac{1}{2}} e^{-\frac{a^2}{4t}}$	$e^{-a\sqrt{s}}$
$(\pi t)^{-\frac{1}{2}} e^{-\frac{a^2}{4t}}$	$\frac{1}{\sqrt{s}} e^{-a\sqrt{s}}$
$1-\mathcal{E}rf\, \frac{a}{\sqrt{4t}}$	$\frac{1}{s} e^{-a\sqrt{s}}$

$$\mathcal{E}rf\ x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Let $F(s)$ and $G(s)$ be the Laplace Transforms of $f(t)$ and $g(t)$.

Function	Transform
$af(t) + bg(t)$	$aF(s) + bG(s)$
$\frac{df}{dt}$	$sF(s) - f(0)$
$\frac{d^2f}{dt^2}$	$s^2F(s) - s f(0) - f'(0)$
$e^{bt} f(t)$	$F(s-b)$
$\frac{f(t)}{t}$	$\int_s^\infty F(s')ds'$
$tf(t)$	$-\frac{dF}{ds}$
$H(t-b) f(t-b)$	$e^{-bs}F(s)$
$f(ct)$	$\frac{1}{c}F(\frac{s}{c})$
$\int_0^t g(t-t')f(t')dt'$	$F(s)G(s)$

Example 1: $\int_0^{+\infty} \frac{\sin(t)}{t} e^{-st}\, dt = \int_s^{+\infty} \frac{ds'}{1+s'^2} = \arctan s' \Big|_s^{+\infty}$

$= \frac{\pi}{2} - \arctan s = \arctan \frac{1}{s}$

Inversion formula $f(t) = \int_{\alpha - i\infty}^{\alpha + i\infty} e^{st} F(s) \frac{ds}{2\pi i}$

This is an integral over the vertical line $s = \alpha + i\beta$ in the complex plane where $-\infty < \beta < +\infty$.

Example 2: $\begin{cases} u_{tt} + \omega^2 u = f(t) \\ u(0) = u'(0) = 0 \end{cases}$

By property (iii), the Laplace transform $U(s)$ satisfies

$s^2 U(s) + \omega^2 U(s) = F(s)$

$\rightarrow \quad U(s) = \frac{F(s)}{s^2 + \omega^2}$

and $u(t) = \int_0^t \frac{1}{\omega} \sin[\omega(t-t')] f(t')\, dt'$

Example 3: $\begin{cases} u_t = k u_{xx} & \text{in } (0,l) \quad t>0 \\ u(0,t) = u(l,t) = 1 \\ u(x,0) = 1 + \sin \frac{\pi x}{l} \end{cases}$

Method 1: Expansion Method

$u(x,t) = 1 + \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}$

Substituting the expansion into the equation, we have

$\sum_{n=1}^{\infty} \left(u_n'(t) \sin \frac{n\pi x}{l} + k u_n(t) \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi}{l} x \right) = 0$

$\rightarrow \quad u_n'(t) + k u_n(t) \left(\frac{n\pi}{l}\right)^2 = 0 \qquad n = 1, 2, \cdots$

$\rightarrow \quad u_n(t) = C_n e^{-k\left(\frac{n\pi}{l}\right)^2 t}$

$u(x,t) = 1 + \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \sin \frac{n\pi x}{l}$

Using the initial condition, we have

$C_n = 0 \quad n \geq 2 \qquad C_1 = e^{k\frac{\pi^2}{l^2} 0} = 1$

$u(x,t) = 1 + e^{-k\frac{\pi^2}{l^2} t} \sin \frac{\pi x}{l}$

Method 2: Laplace Transform

The Laplace Transform of $u(x,t)$: $U(x,s)$ satisfies

$$\begin{cases} s\,U(x,s) - u(x,0) = k\,U_{xx}(x,s) \\ U(0,s) = U(l,s) = 1/s \end{cases}$$

$$U(x,s) = \frac{1}{s} + \frac{1}{s + k\frac{\pi^2}{l^2}} \sin\frac{\pi}{l}x$$

Inverse Laplace Transform gives us

$$u(x,t) = 1 + e^{-k\frac{\pi^2}{l^2}t} \sin\frac{\pi}{l}x$$