§ 12.4 Source Functions

Fourier Transform can be used in finding the source function of a PDE from scratch

Diffusion equation:

The source function is defined as the unique solution of

$$\begin{cases} S_t = S_{xx} & -\infty < x < \infty, \quad t > 0 \\ S(x,0) = \delta(x) \end{cases}$$

where the diffusion constant is 1. Let's assume no knowledge at all about the form of $S(x,t)$. We only assume it has a Fourier Transform as a distribution in x, for each t.

$$\hat{S}(k,t) = \int_{-\infty}^{+\infty} S(x,t)\, e^{-ikx}\, dx .$$

Here k denotes the frequency variable.

By property (i) of Fourier transforms, the PDE takes the form

$$\begin{cases} \dfrac{\partial \hat{S}}{\partial t} = (ik)^2 \hat{S} = -k^2 \hat{S} \\ \hat{S}(k,0) = 1 \end{cases}$$

For each k, this is an ODE with the solution $\hat{S}(k,t) = e^{-k^2 t}$

Then $$S(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-k^2 t} e^{ikx}\, dk = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}$$

which agrees with § 2.4

Wave equation:

The source function for the 1D wave equation satisfies

$$\begin{cases} S_{tt} = c^2 S_{xx} \\ S(x,0) = 0 \\ S_t(x,0) = \delta(x) \end{cases}$$

Using Fourier Transform, we get

$$\begin{cases} \dfrac{\partial^2 \hat{S}(k,t)}{\partial t^2} = -c^2 k^2 \hat{S}(k,t) \\ \hat{S}(k,0) = 0 \\ \dfrac{\partial \hat{S}}{\partial t}(k,0) = 1 \end{cases} \quad \to \hat{S}(k,t) = \frac{1}{kc}\sin kct$$

$$\hat{S}(k,t) = \frac{e^{ikct} - e^{-ikct}}{2ikc}$$

$$S(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{(e^{ikct} - e^{-ikct})}{2ikc} e^{ikx}\, dk$$

According to the table the transform of $sgn(x) = H(x) - H(-x)$ is $\frac{2}{ik}$. By property (iii), the transform of $\frac{sgn(x+a)}{4c}$ is $\frac{e^{iak}}{2ikc}$

$$S(x,t) = \frac{sgn(x+ct) - sgn(x-ct)}{4c} = \begin{cases} 0 & \text{for } |x| > ct \\ \frac{1}{2c} & |x| < ct \end{cases}$$

$$S(x,t) = \frac{H(c^2t^2 - x^2)}{2c}$$

In 3D, the source function has a Fourier Transform $\hat{S}(k,t)$

$$\begin{cases} \frac{\partial^2}{\partial t^2}\hat{S} = -c^2(k_1^2 + k_2^2 + k_3^2)\hat{S} \\ \hat{S}(\vec{k},0) = 0 \\ \frac{\partial \hat{S}}{\partial t}(\vec{k},0) = 1 \end{cases}$$

where $\vec{k} = (k_1, k_2, k_3)$, $k^2 = |\vec{k}|^2 = (k_1^2 + k_2^2 + k_3^2)$

$$S(\vec{x},t) = \iiint_{\mathbb{R}^3} \frac{1}{kc} \sin kct\; e^{i\vec{k}\cdot\vec{x}} \frac{d\vec{k}}{(2\pi)^3}$$

Spherical coordinates $\vec{k}\cdot\vec{x} = kr\cos\theta$

$$S(\vec{x},t) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{kc} \sin kct\; e^{ikr\cos\theta} k^2 \sin\theta \frac{dk\, d\theta\, d\phi}{8\pi^3}$$

$$= \frac{1}{2\pi^2 cr}\int_0^{\infty} \sin kct \sin kr\, dk$$

$$= \frac{1}{8\pi^2 cr}\int_{-\infty}^{+\infty} \left[e^{ik(ct-r)} - e^{ik(ct+r)}\right] dk$$

$$= \frac{1}{4\pi cr}\left[\delta(ct-r) - \delta(ct+r)\right]$$

Notice how the characteristic variables show up again!

For $t > 0$, $ct + r > 0$

$$S(\vec{x},t) = \frac{1}{4\pi cr}\delta(ct - r) = \frac{1}{4\pi c^2 t}\delta(ct - |\vec{x}|)$$

which agrees with our previous answer for $t > 0$

Laplace's equation in a half-plane

$$\begin{cases} u_{xx} + u_{yy} = 0 & y > 0 \\ u(x,0) = f(x) & y = 0 \end{cases}$$

We cannot transform y variable since $y \geq 0$

Define $\quad U(k,y) = \int_{-\infty}^{+\infty} u(x,y)\, e^{-ikx}\, dx$

then U satisfies the ODE

$$\begin{cases} -k^2 U + U_{yy} = 0 & y > 0 \\ U(k,0) = 1 \end{cases}$$

The solutions are $e^{\pm ky}$. We must reject solutions growing exponentially as $|k| \to +\infty$. So $U(k,y) = e^{-|k|y}$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|k|y}\, e^{ikx}\, dk \qquad \text{(improper integral)}$$

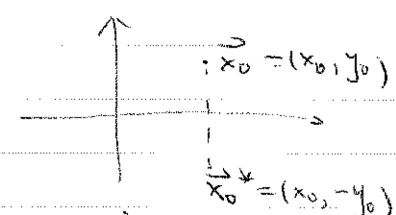
$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{ky} e^{ikx}\, dk + \frac{1}{2\pi}\int_{0}^{+\infty} e^{-ky} e^{ikx}\, dk$$

$$= \frac{1}{2\pi}\,\frac{1}{(ix+y)}\, e^{ikx+ky}\Big|_{-\infty}^{0} + \frac{1}{2\pi}\,\frac{1}{ix-y}\, e^{ikx-ky}\Big|_{0}^{+\infty}$$

$$= \frac{1}{2\pi}\left(\frac{1}{y-ix} + \frac{1}{y+ix}\right) = \frac{y}{\pi(x^2+y^2)}$$

Green's Function for the half-plane

$$G(\vec{x},\vec{x_0}) = \frac{1}{2\pi}\log|\vec{x}-\vec{x_0}| - \frac{1}{2\pi}\log|\vec{x}-\vec{x_0}|$$



satisfies (i) (ii) (iii) properties.

$$G(\vec{x},\vec{x_0}) = \frac{1}{4\pi}\left\{\log\left[(x-x_0)^2+(y-y_0)^2\right] - \log\left[(x-x_0)^2+(y+y_0)^2\right]\right\}$$

$$\frac{\partial G}{\partial n} = \nabla G\cdot n = \begin{pmatrix} \frac{\partial G}{\partial x} \\ \frac{\partial G}{\partial y} \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\frac{\partial G}{\partial y} = \frac{y_0}{\pi\left[(x-x_0)^2+y_0^2\right]} \qquad \text{when } y = 0$$

$$u(x_0,y_0) = \int_{-\infty}^{+\infty} u(x,0)\,\frac{\partial G}{\partial n}\Big|_{y=0} dx = \int_{-\infty}^{+\infty} \delta(x)\,\frac{\partial G}{\partial n}\Big|_{y=0} dx$$

$$= \frac{y_0}{\pi\left[x_0^2+y_0^2\right]}$$

which coincides with the solution obtained by Fourier Transform