§ 12.3 Fourier Transforms

Finite domain ← Fourier series

Whole space ← Fourier Transforms

To understand the relationship, consider $f(x)$ defined on $(-l, l)$. It is fourier series, in complex notation, is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi}{l}x} \quad \text{with } C_n = \frac{1}{2l}\int_{-l}^{l} f(y)\, e^{-\frac{n\pi}{l}y}\, dy$$

What will happen if $l \to +\infty$?

Writing $k = \frac{n\pi}{l}$, we get

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left[\int_{-l}^{l} f(y)\, e^{-iky}\, dy\right] e^{ikx}\, \frac{\pi}{l}$$

From the definition of integration, $\Delta k = \frac{(n+1)\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l}$

When we take the limit (rigorous proof is needed)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(y)\, e^{-iky}\, dy\right] e^{ikx}\, dk$$

Define $F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$, which's the Fourier Transform of $f(x)$

so $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(k)\, e^{ikx}\, dk$$

(x, k) dual variables, k is called the frequency variable

	$f(x)$	$F(k)$
Delta function	$\delta(x)$	1
Square pulse	$H(a-\lvert x\rvert)$	$\frac{2}{k}\sin ak$
Exponential	$e^{-a\lvert x\rvert}$ $(a>0)$	$\frac{2a}{a^2+k^2}$
Heaviside	$H(x)$	$\pi\delta(k) + \frac{1}{ik}$
Sign	$H(x) - H(-x)$	$\frac{2}{ik}$
Constant	1	$2\pi\,\delta(x)$
Gaussian	$e^{-x^2/2}$	$\sqrt{2\pi}\, e^{-k^2/2}$

Properties of Fourier Transform

Let $F(k)$, $G(k)$ be the transform of $f(x)$, $g(x)$.

	Function	Transform
①	$\frac{df}{dx}$	$ik\,F(k)$
②	$x f(x)$	$i\frac{dF}{dk}$
③	$f(x-a)$	$e^{-iak}F(k)$
④	$e^{iax}f(x)$	$F(k-a)$
⑤	$af(x)+bg(x)$	$aF(k)+bG(k)$
⑥	$f(ax)$	$\frac{1}{\lvert a\rvert}F\left(\frac{k}{a}\right)$ $(a\neq 0)$

Parseval's equality

$$\int_{-\infty}^{+\infty} |f(x)|^2\,dx = \int_{-\infty}^{+\infty} |F(k)|^2\,\frac{dk}{2\pi}$$

The Heisenberg Uncertainty Principle

In quantum mechanics, x position variable

k momentum variable

The wave function $f(x)$ satisfies $\int_{-\infty}^{+\infty}|f(x)|^2dx = 1$.

The expected value of the square of the position is

$$\bar{x}^2 = \int_{-\infty}^{+\infty} |x f(x)|^2\,dx$$

The expected value of the square of the momentum is

$$\bar{k}^2 = \int_{-\infty}^{+\infty} |kF(k)|^2\,\frac{dk}{2\pi}$$

The Uncertainty principle asserts that

$$\bar{x}\cdot\bar{k} \geq \frac{1}{2}$$

Remark: Position and momentum can not be precisely determined at the same time.

Proof: By Schwarz's inequality

$$\left|\int_{-\infty}^{+\infty} x\,f(x)\,f'(x)\,dx\right| \leq \left(\int_{-\infty}^{+\infty}|xf(x)|^2dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}|f'(x)|^2dx\right)^{\frac{1}{2}}$$

$$= \bar{x}\left(\int_{-\infty}^{+\infty}|ikF(k)|^2\frac{dk}{2\pi}\right)^{\frac{1}{2}} = \bar{x}\,\bar{k}$$

On the other hand,

$$\int_{-\infty}^{+\infty} x f(x) f'(x) dx = \frac{1}{2} x [f(x)]^2 \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{1}{2} [f(x)]^2 dx = -\frac{1}{2}$$

since $f(x)$ is normalized and $f(x)$ is 0 as $x \to \pm\infty$.

Therefore, $\overline{x} \cdot \overline{k} \geq \frac{1}{2}$

Convolution.

The convolution of $f(x)$ and $g(x)$ is defined as

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y) g(y) dy$$

The Fourier transform

$$\int_{-\infty}^{+\infty} (f*g)(x) e^{-ikx} dx = \iint f(x-y) g(y) e^{-ikx} dx dy$$

$$= \iint f(x-y) e^{-i\cancel{t}(x-y)} g(y) e^{-iky} dx dy$$

Let $z = x - y$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(z) e^{-ikz} g(y) e^{-iky} dz dy$$

$$= \int_{-\infty}^{+\infty} f(z) e^{-ikz} dz \int_{-\infty}^{+\infty} g(y) e^{-iky} dy$$

$$= F(k) \cdot G(k)$$

In 3D. $F(\vec{k}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d\vec{x}$

where $\vec{x} = (x, y, z)$, $\vec{k} = (k_1, k_2, k_3)$ and $\vec{k}\cdot\vec{x} = k_1x_1 + k_2x_2 + k_3x_3$

and

$$f(\vec{x}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\vec{k}) e^{i\vec{k}\cdot\vec{x}} d\vec{k}$$