The test

$$\text{sgn}\,(f(a_n)) \cdot \text{sgn}\,(f(p_n)) > 0 \qquad \text{instead of} \qquad f(a_n) \cdot f(p_n) > 0$$

gives the same result but avoids the possibility of overflow or underflow in the multiplication of $f(a_n)$ and $f(p_n)$.

EXERCISE SET 2.1

1. Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.
2. Let $f(x) = 3(x + 1)(x - \frac{1}{2})(x - 1)$. Use the Bisection method on the following intervals to find p_3.
   a. $[-2, 1.5]$ **b.** $[-1.25, 2.5]$
3. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval.
   a. $[0, 1]$ **b.** $[1, 3.2]$ **c.** $[3.2, 4]$
4. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on each interval.
   a. $[-2, -1]$ **b.** $[0, 2]$ **c.** $[2, 3]$ **d.** $[-1, 0]$
5. Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems.
   a. $x - 2^{-x} = 0$ for $0 \le x \le 1$
   b. $e^x - x^2 + 3x - 2 = 0$ for $0 \le x \le 1$
   c. $2x\cos(2x) - (x + 1)^2 = 0$ for $-3 \le x \le -2$ and $-1 \le x \le 0$
   d. $x\cos x - 2x^2 + 3x - 1 = 0$ for $0.2 \le x \le 0.3$ and $1.2 \le x \le 1.3$
6. Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems.
   a. $3x - e^x = 0$ for $1 \le x \le 2$
   b. $x + 3\cos x - e^x = 0$ for $0 \le x \le 1$
   c. $x^2 - 4x + 4 - \ln x = 0$ for $1 \le x \le 2$ and $2 \le x \le 4$
   d. $x + 1 - 2\sin \pi x = 0$ for $0 \le x \le 0.5$ and $0.5 \le x \le 1$
7. **a.** Sketch the graphs of $y = x$ and $y = 2\sin x$.
   b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = 2\sin x$.
8. **a.** Sketch the graphs of $y = x$ and $y = \tan x$.
   b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = \tan x$.
9. **a.** Sketch the graphs of $y = e^x - 2$ and $y = \cos(e^x - 2)$.
   b. Use the Bisection method to find an approximation to within 10^{-5} to a value in $[0.5, 1.5]$ with $e^x - 2 = \cos(e^x - 2)$.
10. Let $f(x) = (x+2)(x+1)^2x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?
    a. $[-1.5, 2.5]$ **b.** $[-0.5, 2.4]$ **c.** $[-0.5, 3]$ **d.** $[-3, -0.5]$
11. Let $f(x) = (x+2)(x+1)x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?
    a. $[-3, 2.5]$ **b.** $[-2.5, 3]$ **c.** $[-1.75, 1.5]$ **d.** $[-1.5, 1.75]$

12. Find an approximation to $\sqrt{3}$ correct to within 10^{-4} using the Bisection Algorithm. [*Hint:* Consider $f(x) = x^2 - 3$.]

13. Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-4} using the Bisection Algorithm.

14. Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3+x-4=0$ lying in the interval $[1, 4]$. Find an approximation to the root with this degree of accuracy.

15. Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3-x-1=0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

16. Let $f(x) = (x-1)^{10}$, $p = 1$, and $p_n = 1 + 1/n$. Show that $|f(p_n)| < 10^{-3}$ whenever $n > 1$ but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

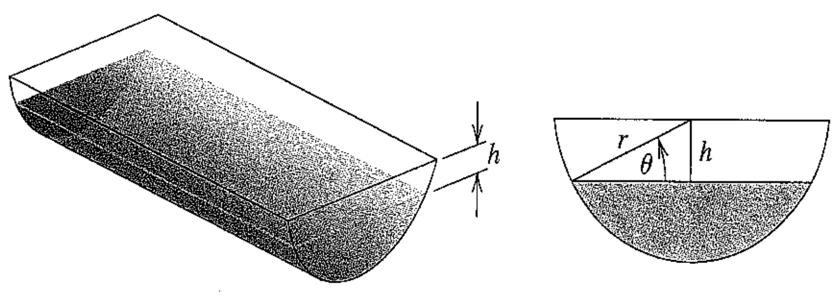
17. Let $\{p_n\}$ be the sequence defined by $p_n = \sum_{k=1}^{n}(1/k)$. Show that $\{p_n\}$ diverges even though $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$.

18. The function defined by $f(x) = \sin \pi x$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the Bisection method converges to

a. 0, if $a+b<2$ **b.** 2, if $a+b>2$ **c.** 1, if $a+b=2$

19. A trough of length L has a cross section in the shape of a semicircle with radius r (see the accompanying figure). When filled with water to within a distance h of the top, the volume V of water is

$$V = L\left[0.5\pi r^2 - r^2 \arcsin\left(\frac{h}{r}\right) - h(r^2-h^2)^{1/2}\right].$$



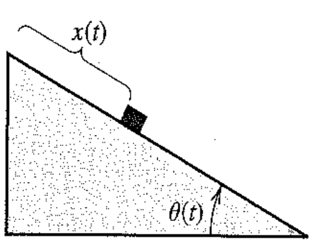
Suppose $L = 10$ ft, $r = 1$ ft, and $V = 12.4$ ft^3. Find the depth of water in the trough to within 0.01 ft.

20. A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$

At the end of t seconds, the position of the object is given by

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right).$$



Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = 32.17$ ft/s^2.

The bound on the magnitude of $g_4'(x)$ is much smaller than the bound (found in (c)) on the magnitude of $g_3'(x)$, which explains the more rapid convergence using g_4.

e. The sequence defined by

$$g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

converges much more rapidly than our other choices. In the next sections we will see where this choice came from and why it is so effective. ■

EXERCISE SET 2.2

1. Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

   a. $g_1(x) = (3 + x - 2x^2)^{1/4}$ **b.** $g_2(x) = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2}$

   c. $g_3(x) = \left(\dfrac{x + 3}{x^2 + 2}\right)^{1/2}$ **d.** $g_4(x) = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

2. **a.** Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.

   b. Which function do you think gives the best approximation to the solution?

3. The following four methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

   a. $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$ **b.** $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

   c. $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$ **d.** $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

4. The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

   a. $p_n = p_{n-1}\left(1 + \dfrac{7 - p_{n-1}^5}{p_{n-1}^2}\right)^3$ **b.** $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{p_{n-1}^2}$

   c. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{5p_{n-1}^4}$ **d.** $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{12}$

5. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on $[1, 2]$. Use $p_0 = 1$.

6. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.

7. Use Theorem 2.2 to show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.4 to estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

8. Use Theorem 2.2 to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-4}. Use Corollary 2.4 to estimate the

number of iterations required to achieve 10^{-4} accuracy, and compare this theoretical estimate to the number actually needed.

9. Use a fixed-point iteration method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 10 of Section 2.1.

10. Use a fixed-point iteration method to find an approximation to $\sqrt[3]{25}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 11 of Section 2.1.

11. For each of the following equations, determine an interval $[a, b]$ on which fixed-point iteration will converge. (i) Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and (ii) perform the calculations.

    a. $x = \dfrac{2 - e^x + x^2}{3}$ **b.** $x = \dfrac{5}{x^2} + 2$ **c.** $x = \left(\dfrac{e^x}{3}\right)^{1/2}$

    d. $x = 5^{-x}$ **e.** $x = 6^{-x}$ **f.** $x = 0.5(\sin x + \cos x)$

12. For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. (i) Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and (ii) perform the calculations.

    a. $2 + \sin x - x = 0$ use $[2, 3]$ **b.** $x^3 - 2x - 5 = 0$ use $[2, 3]$

    c. $3x^2 - e^x = 0$ **d.** $x - \cos x = 0$

13. Find all the zeros of $f(x) = x^2 + 10\cos x$ by using fixed-point iteration for an appropriate iteration function g. Find the zeros accurate to within 10^{-4}.

14. Use a fixed-point iteration method to determine a solution accurate to within 10^{-4} for $x = \tan x$, for x in $[4, 5]$.

15. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $2\sin \pi x + x = 0$ on $[1, 2]$. Use $p_0 = 1$.

16. Let A be a given positive constant and $g(x) = 2x - Ax^2$.

    a. Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = 1/A$, so the reciprocal of a number can be found using only multiplications and subtractions.

    b. Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

17. Find a function g defined on $[0, 1]$ that satisfies none of the hypotheses of Theorem 2.2 but still has a unique fixed point on $[0, 1]$.

18. **a.** Show that Theorem 2.2 is true if the inequality $|g'(x)| \leq k$ is replaced by $g'(x) \leq k$, for all $x \in (a, b)$. [*Hint:* Only uniqueness is in question.]

    b. Show that Theorem 2.3 may not hold if inequality $|g'(x)| \leq k$ is replaced by $g'(x) \leq k$. [*Hint:* Show that $g(x) = 1 - x^2$, for x in $[0, 1]$, provides a counterexample.]

19. **a.** Use Theorem 2.3 to show that the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad \text{for } n \geq 1,$$

    converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

    b. Use the fact that $0 < (x_0 - \sqrt{2})^2$ whenever $x_0 \neq \sqrt{2}$ to show that if $0 < x_0 < \sqrt{2}$, then $x_1 > \sqrt{2}$.

    c. Use the results of parts (a) and (b) to show that the sequence in (a) converges to $\sqrt{2}$ whenever $x_0 > 0$.

20. **a.** Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to $\sqrt{A}$ whenever $x_0 > 0$.

b. What happens if $x_0 < 0$?

21. Replace the assumption in Theorem 2.3 that "a positive number $k < 1$ exists with $|g'(x)| \leq k$" with "g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$." (See Exercise 25, Section 1.1.) Show that the conclusions of this theorem are still valid.

22. Suppose that g is continuously differentiable on some interval (c, d) that contains the fixed point p of g. Show that if $|g'(p)| < 1$, then there exists a $\delta > 0$ such that if $|p_0 - p| \leq \delta$, then the fixed-point iteration converges.

23. An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17$ ft/s^2 and k represents the coefficient of air resistance in lb-s/ft. Suppose $s_0 = 300$ ft, $m = 0.25$ lb, and $k = 0.1$ lb-s/ft. Find, to within 0.01 s, the time it takes this quarter-pounder to hit the ground.

24. Let $g \in C^1[a, b]$ and p be in (a, b) with $g(p) = p$ and $|g'(p)| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.

2.3 Newton's Method

Isaac Newton (1642–1727) was one of the most brilliant scientists of all time. The late 17th century was a vibrant period for science and mathematics and Newton's work touched nearly every aspect of mathematics. His method for solving equations was introduced to find a root of $x^3 - 2x - 5 = 0$, a problem we consider in Exercise 5(a). Although he demonstrated the method only for polynomials, it is clear that he realized its broader applications.

Newton's (or the *Newton–Raphson*) **method** is one of the most powerful and well-known numerical methods for solving a root-finding problem. There are many ways of introducing Newton's method.

If we only want an algorithm, we can consider the technique graphically, as is often done in calculus. Another possibility is to derive Newton's method as a technique to obtain faster convergence than offered by other types of functional iteration, as is done in Section 2.4. A third means of introducing Newton's method, discussed next, is based on Taylor polynomials.

Suppose that $f \in C^2[a, b]$. Let $p_0 \in [a, b]$ be an approximation to the solution p of $f(x) = 0$ such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small." Consider the first Taylor polynomial for $f(x)$ expanded about p_0, and evaluated at $x = p$,

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2}f''(\xi(p)),$$

where $\xi(p)$ lies between p and p_0. Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2}f''(\xi(p)).$$

EXERCISE SET 2.4

1. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.
   a. $x^2 - 2xe^{-x} + e^{-2x} = 0, \quad \text{for } 0 \le x \le 1$
   b. $\cos(x + \sqrt{2}) + x(x/2 + \sqrt{2}) = 0, \quad \text{for } -2 \le x \le -1$
   c. $x^3 - 3x^2(2^{-x}) + 3x(4^{-x}) - 8^{-x} = 0, \quad \text{for } 0 \le x \le 1$
   d. $e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3 = 0, \quad \text{for } -1 \le x \le 0$
2. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.
   a. $1 - 4x\cos x + 2x^2 + \cos 2x = 0, \quad \text{for } 0 \le x \le 1$
   b. $x^2 + 6x^5 + 9x^4 - 2x^3 - 6x^2 + 1 = 0, \quad \text{for } -3 \le x \le -2$
   c. $\sin 3x + 3e^{-2x}\sin x - 3e^{-x}\sin 2x - e^{-3x} = 0, \quad \text{for } 3 \le x \le 4$
   d. $e^{3x} - 27x^6 + 27x^4e^x - 9x^2e^{2x} = 0, \quad \text{for } 3 \le x \le 5$
3. Repeat Exercise 1 using the modified Newton–Raphson method described in Eq. (2.11). Is there an improvement in speed or accuracy over Exercise 1?
4. Repeat Exercise 2 using the modified Newton–Raphson method described in Eq. (2.11). Is there an improvement in speed or accuracy over Exercise 2?
5. Use Newton's method and the modified Newton–Raphson method described in Eq. (2.11) to find a solution accurate to within 10^{-5} to the problem

$$e^{6x} + 1.441e^{2x} - 2.079e^{4x} - 0.3330 = 0, \quad \text{for } -1 \le x \le 0.$$

   This is the same problem as 1(d) with the coefficients replaced by their four-digit approximations. Compare the solutions to the results in 1(d) and 2(d).
6. Show that the following sequences converge linearly to $p = 0$. How large must n be before we have $|p_n - p| \le 5 \times 10^{-2}$?
   a. $p_n = \dfrac{1}{n}, \quad n \ge 1$
   b. $p_n = \dfrac{1}{n^2}, \quad n \ge 1$
7. a. Show that for any positive integer k, the sequence defined by $p_n = 1/n^k$ converges linearly to $p = 0$.
   b. For each pair of integers k and m, determine a number N for which $1/N^k < 10^{-m}$.
8. a. Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to 0.
   b. Show that the sequence $p_n = 10^{-n^k}$ does not converge to 0 quadratically, regardless of the size of the exponent $k > 1$.
9. a. Construct a sequence that converges to 0 of order 3.
   b. Suppose $\alpha > 1$. Construct a sequence that converges to 0 of order α.
10. Suppose p is a zero of multiplicity m of f, where $f^{(m)}$ is continuous on an open interval containing p. Show that the following fixed-point method has $g'(p) = 0$:

$$g(x) = x - \frac{mf(x)}{f'(x)}.$$

11. Show that the Bisection Algorithm 2.1 gives a sequence with an error bound that converges linearly to 0.
12. Suppose that f has m continuous derivatives. Modify the proof of Theorem 2.10 to show that f has a zero of multiplicity m at p if and only if

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p), \quad \text{but } f^{(m)}(p) \ne 0.$$

13. The iterative method to solve $f(x) = 0$, given by the fixed-point method $g(x) = x$, where

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} - \frac{f''(p_{n-1})}{2f'(p_{n-1})}\left[\frac{f(p_{n-1})}{f'(p_{n-1})}\right]^2, \quad \text{for } n = 1, 2, 3, \ldots,$$

has $g'(p) = g''(p) = 0$. This will generally yield cubic ($\alpha = 3$) convergence. Expand the analysis of Example 1 to compare quadratic and cubic convergence.

14. It can be shown (see, for example, [DaB, pp. 228–229]) that if $\{p_n\}_{n=0}^{\infty}$ are convergent Secant method approximations to p, which is the solution to $f(x) = 0$, then a constant C exists such that $|p_{n+1} - p| \approx C|p_n - p||p_{n-1} - p|$ for sufficiently large values of n. Assume $\{p_n\}$ converges to p of order α, and show that $\alpha = (1 + \sqrt{5})/2$. (*Note:* This implies that the order of convergence of the Secant method is approximately 1.62).

2.5 Accelerating Convergence

Theorem 2.7 implies that it is rare to have the luxury of quadratic convergence. We now consider a technique called **Aitken's Δ^2 method** that can be used to accelerate the convergence of a sequence that is linearly convergent, regardless of its origin or application.

Suppose $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p. To motivate the construction of a sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ that converges more rapidly to p than does $\{p_n\}_{n=0}^{\infty}$, let us first assume that the signs of $p_n - p$, $p_{n+1} - p$, and $p_{n+2} - p$ agree and that n is sufficiently large that

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}.$$

Then

$$(p_{n+1} - p)^2 \approx (p_{n+2} - p)(p_n - p),$$

so

$$p_{n+1}^2 - 2p_{n+1}p + p^2 \approx p_{n+2}p_n - (p_n + p_{n+2})p + p^2$$

and

$$(p_{n+2} + p_n - 2p_{n+1})p \approx p_{n+2}p_n - p_{n+1}^2.$$

Solving for p gives

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

Adding and subtracting the terms p_n^2 and $2p_n p_{n+1}$ in the numerator and grouping terms appropriately gives

$$\begin{aligned} p &\approx \frac{p_n p_{n+2} - 2p_n p_{n+1} + p_n^2 - p_{n+1}^2 + 2p_n p_{n+1} - p_n^2}{p_{n+2} - 2p_{n+1} + p_n} \\ &= \frac{p_n(p_{n+2} - 2p_{n+1} + p_n) - (p_{n+1}^2 - 2p_n p_{n+1} + p_n^2)}{p_{n+2} - 2p_{n+1} + p_n} = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}. \end{aligned}$$