§2.3 Newton's Method

$$g(x) = x - \frac{f(x)}{f'(x)} \quad \text{and} \quad x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$\rightarrow$ Newton's method is a FPI.

If $f(x^*) = 0$ & $f'(x^*) \neq 0 \Rightarrow g'(x^*) = 0$

and $|x_{n+1} - x^*|$

$$g(x_n) = g(x^*) + \frac{1}{2} g''(\xi_n)(x_n - x^*)^2 \qquad \text{Taylor's expansion}$$

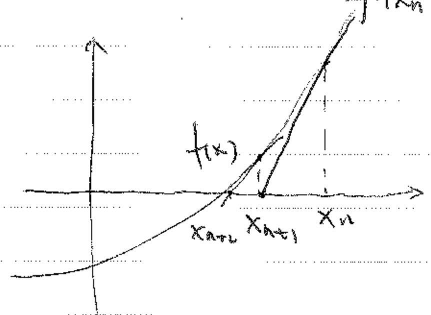
$$= |g(x_n) - x^*| = \frac{1}{2}|g''(\xi_n)|\,|x_n - x^*|^2$$

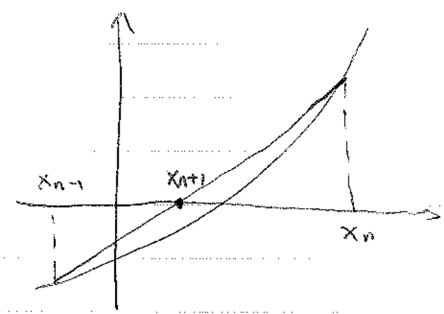
$\Rightarrow$ Quadratic convergence

We can interpret Newton's method as follows.

Approximate $f(x) = 0$ by a simple equation such as $h(x) = 0$ as $h(x) = f(a) + f'(a)(x-a)$ (tangent line). Solve $h(x) = 0 \Rightarrow x = a - \frac{f(a)}{f'(a)}$. Take $a = x_n$, and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$





Secant Method.

We can approximate $f(x) = 0$ by a different equation.

Given x_n, x_{n-1}, define $h(x) = f(x_n) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_n)$

and solve $h(x) = 0$ to get

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

Remark: We can understand this method as a way to avoid computing $f'(x_n)$ by ~~using~~ divided differences

$$\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Method of false position (Regula Falsi)

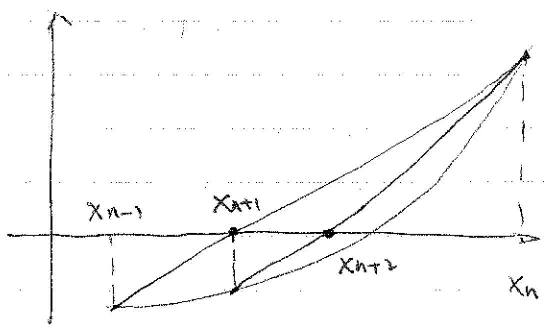
combines the secant and bisection method

Given $(a, f(a))$ and $(b, f(b))$ s.t. $f(a)f(b) < 0$

Construct $x = a - f(a)\dfrac{(a-b)}{f(a)-f(b)}$

If $f(x)f(a) < 0$ take $b = x$ and repeat

Otherwise take $a = x$ and repeat



Remark: The method converges with order $\frac{1+\sqrt{5}}{2} \simeq 0.618\cdots$ (Golden ratio)