§ 2.2 Fixed Point Iteration

We look for p, s.t. $g(p) = p$. The fixed-point iteration (FPI) is: given x_0, define $x_{n+1} = g(x_n)$ $\forall n \geq 0$

If $\{x_n\}_{n=0}^{\infty}$ converges to p, and g is a continuous function, then $p = g(p)$.

Remark: We can always rewrite $f(x)=0$ as a fixed point problem by choosing $g(x) = f(x) + x$, for example (which might not be a good choice)

Ex: Consider $f(x) = x^3 - 3x^2 + 1$

① $g_1(x) = 3 - \frac{1}{x^2}$ $\quad g_1(x) = x \iff 3 - \frac{1}{x^2} = x \iff 3x^2 - 1 = x^3$

② $g_2(x) = \sqrt[3]{3x^2-1}$ $\quad g_2(x) = x \iff \sqrt[3]{3x^2-1} = x \iff 3x^2 - 1 = x^3$

③ $g_3(x) = \frac{x^3+1}{3x}$ $\quad g_3(x) = x \iff \frac{x^3+1}{3x} = x \iff x^3 + 1 = 3x^2$

④ $g_4(x) = x - \frac{x^3 - 3x^2 + 1}{3x^2 - 6x} = x - \frac{f(x)}{f'(x)}$

$$g_4(x) = x \iff x - \frac{f(x)}{f'(x)} = x \iff f(x) = 0$$

We see that some iterations converge, others don't and the speed of convergence is different. Now the question is Can we understand this mathematically?

Assume that $x_n \to x^*$, where $g(x^*) = x^*$, where $x_n = g(x_{n-1})$ and x_0 is given.

then $e_{n+1} := |x_{n+1} - x^*| = |g(x_n) - g(x^*)|$

(by MVT) $= |g'(\xi_n)| \, |x_n - x^*| = |g'(\xi_n)| \, e_n$

where ξ_n is b/n x_n and x^*

Theorem: Consider $g: [a,b] \to [a,b]$ continuous and differentiable and assume that $|g'(x)| \leq k < 1$ $\forall x \in [a,b]$

(or $a \leq g(x) \leq b$, $\forall x \in [a,b]$)

Given $x_0 \in (a,b)$, define $x_{n+1} = g(x_n)$. Then $\{x_n\}$ converges to a fixed point of g, x^*, and moreover x^* is the only fixed point of g in $[a,b]$.

Proof: We first prove the uniqueness.

Assume $z_1, z_2 \in [a,b]$ are fixed points, then

$z_j = g(z_j) \quad j = 1, 2$ and

$$|z_1 - z_2| = |g(z_1) - g(z_2)| = |g'(\xi)| |z_1 - z_2| \leq K |z_1 - z_2|$$

This only happens if $z_1 = z_2$. #

We now show the existence of x^*, by proving that $\{x_n\}$ converges since it is a Cauchy sequence.

$$|x_{n+1} - x_n| = |g(x_n) - g(x_{n-1})| = |g'(\xi_n)| |x_n - x_{n-1}|$$
$$\leq K |x_n - x_{n-1}| \leq K^2 |x_{n-1} - x_{n-2}| \leq \cdots \leq K^n |x_1 - x_0|$$

then

$$|x_n - x_m| \underset{n>m}{\leq} |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$$
$$\leq K^{n-1}|x_1 - x_0| + K^{n-2}|x_1 - x_0| + \cdots + K^m |x_1 - x_0|$$
$$= (K^{n-1} + K^{n-2} + \cdots + K^m) |x_1 - x_0|$$
$$= K^m (1 + K + \cdots + K^{n-m-1}) |x_1 - x_0|$$
$$(0<K<1) \leq \frac{K^m}{1-K} |x_1 - x_0| \longrightarrow 0 \quad \text{as } m \to +\infty$$

$\Rightarrow \{x_n\}$ is Cauchy sequence, so $\exists x^* \in [a,b]$, and $x_n \to x^*$

Since $x_{n+1} = g(x_n)$ and g is continuous

$\Rightarrow \quad x^* = g(x^*)$ #

Note: i) By taking $n \to \infty$ we get the estimate

$$|x^* - x_m| \leq \frac{K^m}{1-K} |x_1 - x_0|$$

ii) $|x_{n+1} - x^*| = |g(x_n) - g(x^*)| \leq K |x_n - x^*|$

So $\{x_n\}_{n=0}^{\infty}$ generated by the FPI algorithm, converges to x^* of order 1, with asymptotic error cost K (upper bound)