Solutions of equations in one variable

Example: $f(x) = ae^{x} + b(e^{x}-1) - N_0$

$f(x) = 0 \Rightarrow x = \ln\left(\frac{N_0+b}{a+b}\right)$ is the root of $f(x)$.

$\frac{df(x)}{dx} = f(x) + b$ describes the exponential model with the constant immigration rate b.

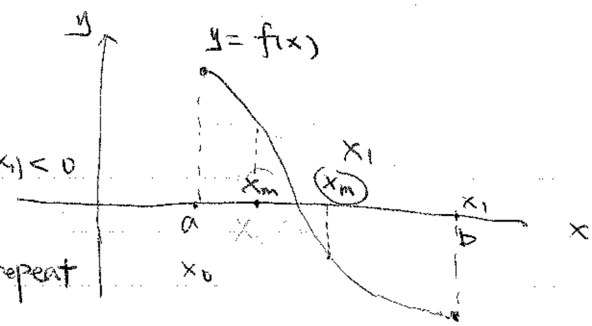
§ 2.1 The Bisection Method

We want to solve the equation $f(x)=0$ (root-finding problem)

Theory: The Intermediate Value Theorem

Suppose $f(x) \in C([a,b])$, with $f(a)$ & $f(b)$ of opposite sign then there exists a number $p \in (a,b)$ s.t. $f(p)=0$

Method: Bisection



Define $x_0 = a$, $x_1 = b$

then $x_m = \frac{x_0+x_1}{2}$ & $f(x_0)f(x_1) < 0$

If $f(x_0)\cdot f(x_m) \leq 0$ then set $x_1 = x_m$, otherwise $x_0 = x_m$ } and repeat

Termination: We know that there is always a root in the interval $[x_0, x_1]$, so we repeat the procedure until the length of the interval is sufficiently small.

Pseudocode:

INPUT endpoints a, b; tolerance TOL; maximum number of iterations N_0; function $f(x)$.

OUTPUT approximate solution p or message of failure.

Step 1 Set $j=1$; $x_0=a$; $x_1=b$;
$f_0 = f(a)$; $f_1 = f(b)$.

Step 2 While $j \leq N_0$ do Steps 3 - 6.

Step 3 Set $x_m = \frac{x_0+x_1}{2}$; $f_m = f(x_m)$.

Step 4 If $\frac{(x_1-x_0)}{2} < TOL$ then
OUTPUT (p); STOP.

Step 5 Set $j = j+1$

Step 6 If $f_0 \cdot f_m \leq 0$, then

set $x_1 = x_m$; $f_1 = f_m$;

else set $x_0 = x_m$, $f_0 = f_m$.

Step 7 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);

STOP.

Implementation: Matlab

Theorem: Given a continuous function $f : [a,b] \to \mathbb{R}$, such that $f(a) \cdot f(b) < 0$; let p_n be the middle point of the n-th interval generated by the bisection algorithm. Then $p_n \to p$ for some $p \in [a,b]$, s.t. $f(p) = 0$ and

$$|p_n - p| \leq \frac{b-a}{2^n} \qquad \forall n \in \mathbb{N} \cup \{0\}$$

Proof. The Bisection algorithm generates a sequence of intervals $I_n = [a_n, b_n]$, s.t.

i) $f(a_n) f(b_n) \leq 0$

ii) $I_{n+1} \subset I_n \quad \forall n$, and $p_n \in I_n$

iii) $\text{length}(I_n) = b_n - a_n = \frac{b-a}{2^n}$

$\forall \varepsilon > 0$, $\exists n_0 > 0$, s.t. $\frac{b-a}{2^n} < \varepsilon \quad \forall n \geq n_0$

$\Longrightarrow \quad |p_n - p_m| < \frac{b-a}{2^{n_0}} < \varepsilon \quad \forall m, n \geq n_0$

so $\{p_n\}$ is a Cauchy sequence and therefore $\exists p \in [a,b]$, s.t. $p_n \to p$. It follows that $p \in I_n$, $\forall n \in \mathbb{N}$

$$|p_n - p| \leq \frac{b-a}{2^n}$$

Since $|a_n - p| \leq \frac{b-a}{2^n}$ & $|b_n - p| \leq \frac{b-a}{2^n}$, then $a_n \to p$, $b_n \to p$ and f is continuous. $f(a_n) f(b_n) \leq 0$

$f(p) f(p) \leq 0 \longrightarrow f(p) = 0$ #

Def: Suppose $\{p_n\}_{n=1}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{|p_n - p|^{\alpha}} = \lambda, \text{ then}$$

$\{p_n\}_{n=1}^{\infty}$ converges to p of order α, with asymptotic error const λ.

Moreover, if $\alpha = 1$, the sequence is linearly convergent; if $\alpha = 2$, the sequence is quadratically convergent.

Remark: $\{p_n\}_{n=1}^{\infty}$ generated by the Bisection algorithm converges to p of order 1, with asymptotic error constant $\frac{1}{2}$.