§ Algorithms and Convergence

$\begin{cases} \text{Round-off error: finite digits, e.g., } \sqrt{3}, \frac{1}{3} \\ \text{Approximation error: Taylor expansion} \end{cases}$

Def: An algorithm is a procedure that describes a finite sequence of steps to be performed in a specified order.

Note: Pseudo codes will be used to describe algorithms, which specifies the form of the input to be supplied and the form of the desired output.

Def: Given $\{\beta_n\}$, $\{\alpha_n\}$, $\beta_n \to 0$, $\alpha_n \to \alpha$. We say that α_n converges to α with the rate of convergence $O(\beta_n)$ (big O) if $\exists K > 0$, s.t.

$$|\alpha_n - \alpha| \le K|\beta_n| \quad n \to \infty \quad \text{or} \left|\frac{\alpha_n - \alpha}{\beta_n}\right| \le K$$

and we write $\alpha_n = \alpha + O(\beta_n)$

If $\lim_{n\to\infty} \left|\frac{\alpha_n - \alpha}{\beta_n}\right| = 0$, we say that α_n converges to α with the rate of convergence $o(\beta_n)$ (little o) and we write $\alpha_n = \alpha + o(\beta_n)$

Remark: $\alpha_n = \alpha + o(\beta_n)$ implies $\alpha_n = \alpha + O(\beta_n)$

Ex: Usually we will take $\beta_n \sim \frac{1}{n^p}$

Consider $\alpha_n = e^{\frac{1}{n}}$ $\quad \lim_{n\to\infty} \alpha_n = 1$

We want to know how fast $\alpha_n \to 1$

By Taylor's expansion, $e^{\frac{1}{n}} = 1 + \frac{1}{n} + \frac{1}{2!}\left(\frac{1}{n}\right)^2 + \frac{1}{3!}\left(\frac{1}{n}\right)^3 + \cdots$

$\Rightarrow \lim_{n\to\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = 1$ and $e^{\frac{1}{n}} = 1 + O\left(\frac{1}{n}\right)$

Moreover, if $b_n = e^{\frac{1}{n}} - 1 - \frac{1}{n} - \frac{1}{2}\frac{1}{n^2}$ $\quad \lim_{n\to\infty} b_n = 0$

What is the rate of convergence?

$e^{\frac{1}{n}} = 1 + \frac{1}{n} + \frac{1}{2}\frac{1}{n^2} + O\left(\frac{1}{n^3}\right)$

Ex: $a_n = \cos\left(\frac{1}{n}\right)$ $\quad a_n \xrightarrow{n\to\infty} 1$

$$\lim_{n\to\infty} \frac{a_n - 1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right) - 1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{2n^2} + \cdots}{\frac{1}{n}} = 0$$

since $\displaystyle \cos x \underset{x=0}{=} 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n - 1}{\frac{1}{n^2}} = -\frac{1}{2} \Rightarrow a_n = 1 + O\left(\frac{1}{n^2}\right)$$

$$a_n = 1 + o\left(\frac{1}{n}\right)$$

Def: Given $F(h)$, $G(h)$ functions such that $F(h) \to L$ as $h \to 0$ and $G(h) \to 0$ as $h \to 0$. We say that $F(h) = L + O(G(h))$ if

$$\left| \frac{F(h) - L}{G(h)} \right| \leq K \quad \text{as } h \to 0 \qquad K > 0$$

If $\displaystyle \lim_{h\to 0} \left| \frac{F(h) - L}{G(h)} \right| = 0 \Rightarrow F(h) = L + o(G(h))$

Ex: $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + O(x^6)$

Note: ① If $\begin{cases} F_1(h) = L_1 + O(G_1(h)) \\ F_2(h) = L_2 + O(G_2(h)) \end{cases}$ and $G_1(h) = o(G_2(h))$

$$\Rightarrow F_1(h) + F_2(h) = L_1 + L_2 + O(G_2(h))$$

Ex: $\begin{cases} \cos x = 1 + O(x^2) \\ e^x = 1 + O(x) \end{cases}$ $\quad e^x + \cos x = 2 + O(x)$

② $\begin{cases} F_1(h) = L_1 + O(G(h)) \\ F_2(h) = L_2 + O(G(h)) \end{cases}$, then $F_1(h) - F_2(h) = L_1 - L_2 + O(G(h))$

Ex: $\cos h = 1 + O(h^2) = 1 - \frac{h^2}{2} + O(h^4)$

$e^{h^2} = 1 + O(h^2) = 1 + h^2 + O(h^4)$

$e^{h^2} - \cos h = \frac{3}{2}h^2 + O(h^4) = O(h^2)$

③ If $F(h) = L + O(G_1(h))$ $(G_1(h) \to 0)$ and $G_2(h) \to 0$

then $F(h) \cdot G_2(h) = O(G_2(h))$ if $L \neq 0$

If $L = 0 \to F(h) \cdot G_2(h) = O(G_1(h) \cdot G_2(h))$

Ex: $\cos x = 1 - \frac{x^2}{2} + O(x^4)$

$x^2 \cos x = x^2\left(1 - \frac{x^2}{2} + O(x^4)\right) = x^2 - \frac{x^4}{2} + x^2 O(x^4)$

$= O(x^2)$

$\sin x = x + O(x^3)$ $\qquad x^2 \sin x = O(x^3)$