Review of Calculus

Def: (Limit of a function)

Given $f: I \to \mathbb{R}$ (I interval) and c a (accumulation) point of I, we say the limit of $f(x)$ as x approaches c is L, or $\lim_{x \to c} f(x) = L$

if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $|f(x) - L| < \varepsilon$
$0 < |x - c| < \delta$,

Def: (Continuity)

Given $f: I \to \mathbb{R}$ ($\mathbb{C}$) and $x_0 \in I$, we say that f is continuous at x_0 if

$\forall \varepsilon > 0$ $\exists \delta > 0$, $|x - x_0| < \delta$, s.t. $|f(x) - f(x_0)| < \varepsilon$

Def: (Convergence of a sequence)

Consider an infinite sequence of real numbers $\{x_n\}_{n=1}^{\infty}$

We say that $\{x_n\}$ converges to $L \in \mathbb{R}$, or $\lim_{n \to \infty} x_n = L$ if

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $|x_n - L| < \varepsilon$, $\forall n \geq n_0$

Theorem: Given $f: I \to \mathbb{R}$, then f is continuous at x_0 if and only if $\forall \{x_n\}_{n=1}^{\infty} \subseteq I$, s.t. $x_n \to x_0$,

we have $f(x_n) \to f(x_0)$

Proof: "$\Rightarrow$" Assume f is continuous at x_0, and consider $\{x_n\}_{n=1}^{\infty} \subseteq I$, s.t. $x_n \to x_0$

Given $\varepsilon > 0$, $\exists \delta > 0$: if $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

For the same $\delta > 0$ $\exists n_0 \in \mathbb{N}$: $|x_n - x_0| < \delta$ if $n > n_0$

$\Rightarrow \exists n_0 \in \mathbb{N}$: $|f(x_n) - f(x_0)| < \varepsilon$ if $n \geq n_0$

So $\lim_{n \to \infty} f(x_n) = f(x_0)$

"$\Leftarrow$" Assume $\forall \{x_n\} \to x_0$, $f(x_n) \to f(x_0)$, and assume that f is not discontinuous at x_0

$\Rightarrow \exists \varepsilon > 0$, $\forall \delta > 0$, $\exists x_\delta \in I$, s.t. $|x_\delta - x_0| < \delta$ and $|f(x_\delta) - f(x_0)| \geq \varepsilon$

Choose $\delta_n = \frac{1}{n} \Rightarrow \exists \{x_n\} \subset I$, s.t. $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon \Rightarrow x_n \to x_0$ and $f(x_n) \not\to f(x_0)$

$\Rightarrow f$ is continuous at x_0

Def: (Uniform continuity)
We say that $f: I \to \mathbb{R}$ is uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $|f(x) - f(y)| < \varepsilon$ if $|x-y| < \delta$

Ex: $I = (0,1)$ $f(x) = \frac{1}{\sqrt{x}}$, then f is continuous over I, but not uniformly continuous

Def: Differentiability
A function $f: I - \mathbb{R}$ is said to be differentiable at $x_0 \in I$ if the following limit exists

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0)}{x - x_0} = L$$

then L is called the derivative of f at x_0, denoted $f'(x_0)$, $\frac{df}{dx}\Big|_{x=x_0}$, $\frac{df}{dx}(x_0)$

Thm: (Heine - Borel Theorem)
If $f: [a,b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.
Note that the domain is closed and bounded (which means compact in $\mathbb{R}^n$)

Thm: If $f: [a,b] \to \mathbb{R}$ is continuous, then f attains its maximum and minimum in $[a,b]$, i.e., $\exists x_1, x_2 \in [a,b]$, s.t., $f(x_2) \le f(x) \le f(x_1)$ $\forall x \in [a,b]$

Thm: If $f: [a,b] \to \mathbb{R}$ is continuous ($\mathcal{C}([a,b];\mathbb{R})$ or $\mathcal{C}([a,b])$) and differentiable on (a,b), then, if X_{max} and X_{min} are the points in $[a,b]$ where the max and min of f are achieved, respectively, then either $X_{max} = a$, $X_{max} = b$ or $f'(X_{max}) = 0$, and similarly for X_{min}.

Thm. (Rolle's Theorem)

If $f: [a,b] \to \mathbb{R}$ is continuous in $[a,b]$ and differentiable in (a,b), then if $f(a) = f(b) \Rightarrow \exists c \in (a,b)$, s.t. $f'(c) = 0$

Proof: By the previous thm. we have

$\begin{cases} \text{either } x_{min} \in (a,b) \text{ or } x_{max} \in (a,b) \\ \text{or } x_{min}, x_{max} \in \{a,b\} \end{cases}$, then

$\begin{cases} \exists c \in (a,b) \quad f'(c) = 0 \qquad c = x_{min} \text{ or } x_{max} \\ f(a) = f(b) \Rightarrow f \text{ is constant, so } f'(c) = 0 \quad \forall c \in (a,b) \end{cases}$

Thm: (Mean Value Theorem)

If $f: [a,b] \to \mathbb{R} \in C([a,b])$ and $C'((a,b))$, then $\exists c \in (a,b)$, s.t., $f(b) - f(a) = f'(c)(b-a)$

Proof: Apply Rolle's Thm to $g(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right)$

Def: (Riemann Integral)

Given a bounded function $f: [a,b] \to \mathbb{R}$ we define a Riemann sum as $R(f; P) = \sum_{i=1}^{n} f(z_i)\Delta x_i$

where P = partition of $[a,b] = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$

$\Delta x_i = x_i - x_{i-1}$, and $z_i \in (x_{i-1}, x_i]$.

We say that f is Riemann-integrable if the limit $\lim_{\max\{\Delta x_i\} \to 0} R(f; P)$ exists, and we call it the Riemann integral, denoted by $\int_a^b f(x)dx$

Thm: Fundamental theorem of Calculus

If $f \in C([a,b])$, then $F(x) = \int_a^x f(t)dt$ is differentiable and $F'(x) = f(x)$, $\forall x \in [a,b]$

Note: If $f \in C([a,b])$, then f is Riemann integrable on $[a,b]$

Thm: Intermediate value Theorem

Given $f: [a,b] \to \mathbb{R}$ continuous, and $f(x_1) \leq k \leq f(x_2)$ for some $x_1, x_2 \Rightarrow \exists x_3 \in [x_1, x_2]: f(x_3) = k$

Thm: Weighted Mean value theorem

If $f \in C([a,b])$ and $\int_a^b g(x)\,dx$ exists for some function g that does not change sign in $[a,b]$ (either $g \geq 0$, or $g \leq 0$), then $\exists\ c \in [a,b]$, s.t.

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

Proof: Let $x_1, x_2 \in [a,b]$, s.t., $m = f(x_1) = \min_{x\in[a,b]} f(x)$

$M = f(x_2) = \max_{x\in[a,b]} f(x)$. Assume $g \geq 0$

then $m g(t) \leq f(t)g(t) \leq M g(t)$, so

$$m \leq \frac{1}{\int_a^b g(t)dt}\int_a^b f(t)g(t)\,dt \leq M \qquad \left(\begin{array}{l}\text{If } \int g = 0 \Rightarrow g \equiv 0 \\ \text{so it is trivial}\end{array}\right)$$

By the intermediate value theorem, $\exists\ c \in [a,b]$, s.t.

$$f(c) = \frac{1}{\int_a^b g(t)dt}\int_a^b f(t)g(t)\,dt$$

Remark: If $g \equiv 1 \Rightarrow \int_a^b f(t)\,dt = f(c)(b-a)$ for some c

Thm. Generalized Rolle's Theorem

If $f \in C([a,b])$ is n-times differentiable in (a,b), then if f is zero at $(n+1)$ points $x_0, \ldots, x_n$ in (a,b), then $\exists\ c \in (a,b)$. s.t. $f^{(n)}(c) = 0$

Proof: Apply Rolle's Theorem repeatedly.

Taylor's Theorem. Given $f \in C^n([a,b])$ (f is n-times differentiable and the n derivatives are continuous), such that f^{n+1} exists. Consider $x_0 \in [a,b]$. Then define

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j \qquad \text{Taylor Polynomial of degree} \leq n$$

Then $\forall\ x \in [a,b]$, $\exists\ \xi_x$ between x_0 and x, s.t.

$$f(x) = P_n(x) + \underbrace{\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)^{n+1}}_{R_n(x)\ \text{remainder}}$$

Note: Also $f(x) = P_n(x) + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt$

Ex: $f(x) = e^x$

$P_n(x) = \sum_{j=0}^{n} \frac{e^{x_0}}{j!}(x-x_0)^j$, $R_n(x) = e^{\xi_x}\frac{(x-x_0)^{n+1}}{(n+1)!}$

$|e^x - P_n(x)| = |R_n(x)| = \frac{e^{\xi_x}}{(n+1)!}|x-x_0|^{n+1}$

So if $|x-x_0| < L$

$$|R_n(x)| \le e^{x_0+L}\frac{L^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$$

Def: Truncation error is the error commited in using the truncated series $E(x) := |f(x) - P_n(x)| = |R_n(x)|$