My research is in the areas of geometric group theory and low dimensional topology. As a geometric group theorist, when given a group, I build a topological space on which the group acts by structure-preserving maps. In this way I can explore algebraic properties of the group by understanding the geometry and topology of the space and visa versa. The particular groups I have been investigating for my thesis are special linear groups over commutative rings with one. In order to study these special linear groups, $SL(n, R)$, I constructed a simplicial complex, $\tau_n(R)$, which I call the n^{th} Torus Complex over R, on which $SL(n, R)$ acts simplicially. I have found some relationships between properties of the ring R and properties of $\tau_n(R)$. I also use the machinery of complexes of groups to calculate group presentations for certain special linear groups. Specifically, there is a theorem of Haefliger [4] which describes how, given a group acting cocompactly without inversions on a path connected, simply connected CW complex, one can deduce a presentation. Special linear groups over rings are of interest to many types of mathematicians. Number theorists study quadratic forms via $SL(2, R)$. Topologists and geometric group theorists are interested in arithmetic groups, which are subgroups of $SL(n, R)$. These groups can give rise to hyperbolic manifolds [9] and orbifolds and are also related to lattices. And algebraists who study K-theory often consider presentations for the subgroup of $SL(n, R)$ generated by elementary matrices.

1 Relationships to Previous Work

While presentations of two-dimensional special linear groups have been studied since the 1970's, when Bass-Serre Theory was invented (the theory of graphs of groups [10]), presentations of many 2-dimensional special linear groups over rings of integers in real quadratic number fields are not known. There are even fewer concrete presentations of higher dimensional special linear groups over rings. In the 1970's Swan [11] used Bass-Serre theory to find presentations for various 2-dimensional special linear groups over rings of integers in imaginary quadratic number fields. He exploited the fact that these special linear groups are isomorphic to subgroups of the group of isometries of hyperbolic space. Soon after, Alperin [2], [1] constructed a contractible simplicial complex on which $SL(2, \mathbb{Z}[\theta])$ acts (where θ is the golden ratio or the third root of unity), finding the cohomology of the group as well as a presentation. The 2nd torus complex over a ring R is the 2-skeleton Alperin's simplicial complex. But as he noticed, his construction yields a contractible complex for only a few rings. Rather than trying to show contractibility, I am attempting to show simple connectivity for a wide variety of rings. Because I have successfully produced presentations isomorphic to those of Swan and Alperin, I am optimistic that I may be able to extend my method to other cases. For higher dimensional special linear groups, my complex is not closely related to any others of which I am aware. In her thesis, Charney [6] studied n-dimensional general linear groups over Dedekind domains by constructing a complex on which they act. She proved that her complex is homotopy equivalent to a wedge of spheres, and used the complex to study homology. However, her construction is different from mine and was inspired by the Tits building for a vector space; the vertices are *pairs* of submodules whose direct sum is R^n, and the complex is the geometric realization of a partial order on those pairs.

In the 1980's, the theory of complexes of groups was developed through the work of Haefliger [4] as well as Armstrong [3] and Brown [5]. In particular, this theory has made calculating the fundamental group of a complex of groups a straight-forward process which I use to derive a new presentation of $SL(3,\mathbb{Z})$ which is palindromic and has finite order generators. I define a group presentation to be palindromic if any relation written backwards is still a relation, and I have shown that not every group has such a presentation. In 1992, Conder et. al. [8] found presentations for $SL(3,\mathbb{Z}_p), SL(3,\mathbb{Z})$ via purely group theoretic means. But these presentations are not palindromic. In the future, I would like to extend my results to both higher dimensions and other rings.

2 Main Results of My Thesis

Given an integer n, I first constructed a simplicial complex $\tau_n(\mathbb{Z})$, called the nth Torus Complex over $\mathbb{Z}$, on which $SL(n,\mathbb{Z})$ acts simplicially. The construction is analogous to that of the curve complex of a 2 dimensional torus, a complex on which $SL(2,\mathbb{Z})$ acts. In fact, $\tau_2(\mathbb{Z})$ is the Curve Complex of the Torus. I then found an equivalent algebraic definition of this complex that allowed me to extend the construction to rings other than $\mathbb{Z}$. The following table show some of the properties of $\tau_n(R)$ that I prove are consequences of properties of the ring R.

n	Properties of R	Properties of $\tau_n(R)$
2	R generated additively by units	link of every vertex is connected
2	$R = \mathbb{Z}[\sqrt{-n}], n = 1, 3$	$\pi_1 = \{1\}$, not contractible for $n = -1$
3	$R = \mathbb{Z}$	connected, $\pi_1 = \{1\}$, diameter 2
n	Euclidean	connected
$n \geq 3$	$R = \mathbb{Z}$	$\pi_{n-2}(\tau_n(\mathbb{Z})) \cong \{1\}$

The n^{th} **Unoriented Torus Complex**, denoted τ_n, is the simplicial complex whose vertices correspond to isotopy classes of essential unoriented co-dimension 1 tori. A $k-1$-simplex of τ_n $(k-1 \leq n)$ is spanned by k vertices if and only if the codimension one tori to which these vertices correspond intersect transversally in exactly one codimension k torus after isotopy. This topological definition can be extended to the following algebraic definition.

The n^{th} **Torus Complex over** R (R a Commutative ring with 1), denoted $\tau_n(R)$, is the simplicial complex whose vertices correspond to rank one free summands L_1 of R^n with the property that there exist $L_k, \quad k = 2, 3, \ldots n$ such that $\bigoplus_{i=1}^n L_i \cong R^n$. A $k-1$-simplex of $\tau_n(R)$ $(k-1 \leq n)$ is spanned by k vertices if and only if the direct sum of the corresponding rank one free summands is again a free summand of R^n.

Theorem 2.1. *$\tau_3(\mathbb{Z})$ is simply connected and $SL(3,\mathbb{Z})$ acts on $\tau_3(\mathbb{Z})$ cocompactly (but not properly). Taking the fundamental group of the complex of groups obtained by labeling cells in the quotient by their stabilizer groups yields the following presentation for $SL(3,\mathbb{Z})$.*

$$\big\langle a, b, c | a^4, b^6, c^2, a^2b^3, (a^2cabc)^2, (cabcb^2)^3, (ac)^3,$$
$$ab(cabc)(ab)^{-1}(cabc)^{-1}, ba(cbac)(ba)^{-1}(cbac)^{-1}\big\rangle$$

More recently, I calculated the fundamental group of the complex of groups obtained from the action of $SL(2,R)$ on $\tau_2(R)$ with R a ring of integers in an imaginary quadratic number field. By consulting Cohn's work [7], one sees that this is isomorphic to $E(2,R)$, the group generated by elementary matrices over R. The complexes of groups associated to $\tau_2(R)$ for R a ring of integers in a *real* quadratic number field seem much different due to the fact that there are infinitely many units in R.

3 Simple Connectivity of Simplicial Complexes

As a result of exploring the relationships between a ring, its special linear groups, and the corresponding torus complexes, I have developed methods by which one can show that the simplicial complexes $\tau_n(R)$ are simply connected. Instead of using typical curve-shortening algorithms, I assign a height to each edges or vertices (depending on n). This height function is defined so that there is one simplex of lowest height. Given a loop, I focus on a highest simplex and find a homotopic loop whose highest simplices are lower. In this process, the loop may lengthen, but its height is lower. By repeating this process, the loop is contracted. I used the same idea of assigning a height function to cells to show that $\pi_{n-2}(\tau_n(\mathbb{Z})) = \{1\}$.

4 Future research questions:

My thesis work can be extended in many directions which range from more algebraic to more topological. The following are some of the questions I hope to answer in the future.

Question 1. Other generalizations of the curve complex suggest themselves. For example, using isotopy classes of essential surfaces in a fixed 3-manifold or using subvarieties of an algebraic variety.

Question 2. What rings are characterized by the property that $\tau_n(R)$ is connected? simply connected? contractible?

Question 3. Is the 2nd Torus Complex over a ring of integers in a *real* quadratic number field simply connected? If so, this might lead to a concrete presentation of $SL(2, \mathbb{Z}[\sqrt{2}])$.

Question 4. What groups are palindromic (the braid groups are an example) and what geometric properties do these groups have?

Question 5. What properties of a height function and the simplicial complex it is labeling guarantee simple connectivity?

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