A heat trace anomaly on polygons

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Abstract

Let Ω_0 be a polygon in $\mathbb{R}^2$, or more generally a compact surface with piecewise smooth boundary and corners. Suppose that Ω_ϵ is a family of surfaces with $\mathcal{C}^\infty$ boundary which converges to Ω_0 smoothly away from the corners, and in a precise way at the vertices to be described in the paper. Both Kac [3] and McKean-Singer [7] recognized that certain heat trace coefficients, in particular the coefficient of t^0, are not continuous as $\epsilon \searrow 0$. We describe this anomaly using renormalized heat invariants of an auxiliary smooth domain Z which models the corner formation.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary, or more generally, any two dimensional compact Riemannian manifold with smooth boundary. The Laplace operator with Dirichlet boundary conditions has discrete spectrum $\{\lambda_i\}$ and corresponding eigenfunctions $\{\phi_i\}$. The fundamental solution for the Cauchy problem for the heat equation has Schwartz kernel

$$H^\Omega(t,z,z') = \sum_{i=1}^{\infty} e^{-\lambda_i t}\phi_i(z)\phi_i(z');$$

this converges in $\mathcal{C}^\infty((0,\infty)\times\overline{\Omega}\times\overline{\Omega})$ and is even smooth up to $t=0$ away from the diagonal of $\Omega\times\Omega$. The so-called heat trace is the function

$$\operatorname{Tr} H^\Omega = \sum_{i=1}^{\infty} e^{-\lambda_i t} = \int_\Omega H^\Omega(t,z,z)\,dz; \tag{1.1}$$

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this has an asymptotic expansion as $t \searrow 0$ of the form

$$\operatorname{Tr} H^\Omega \sim \sum_{j=0}^{\infty} a_j t^{-1+\frac{j}{2}}. \tag{1.2}$$

Each coefficient a_j is a sum of two terms: an integral over Ω of some universal polynomial in the Gauss curvature K of the metric and its covariant derivatives, and an integral over $\partial\Omega$ of another universal polynomial in the geodesic curvature κ of the boundary and its derivatives. Precise formulæ for these polynomials are extremely complicated (and mostly unknown) when j is large, but the first few are quite simple:

$$a_0 = \frac{1}{4\pi}\int_\Omega 1\, dA = \frac{1}{4\pi}|\Omega|, \qquad a_1 = -\frac{1}{8\sqrt{\pi}}\int_{\partial\Omega} 1\, ds = -\frac{1}{8\sqrt{\pi}}|\partial\Omega|$$

and

$$a_2 = \frac{1}{12\pi}\left(\int_\Omega K\, dA + \int_{\partial\Omega} \kappa\, ds\right) = \frac{1}{6}\chi(M). \tag{1.3}$$

Here and elsewhere, $|\cdot|$ refers to either area of a domain or length of its boundary, as appropriate.

Almost all of this remains true if the boundary of Ω is piecewise smooth. More precisely, assume that $\partial\Omega$ is a finite union of smooth arcs, γ_i, $i = 1, \ldots, k$, where (counting indices mod k) γ_i meets γ_{i+1} at the vertex p_i with an interior angle $\alpha_i \in (0, 2\pi)$. In fact, the only modification in the statements above is that the heat trace coefficients may now include contributions from the vertices. The formulæ for a_0 and a_1 are the same as before, but now

$$a_2 = \frac{1}{12\pi}\left(\int_\Omega K\, dA + \sum_{j=1}^{k}\int_{\gamma_j} \kappa\, ds\right) + \sum_{j=1}^{k}\frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j}. \tag{1.4}$$

The term in parentheses on the right now equals $2\pi\chi(\Omega) - \sum_{j=1}^k \alpha_j$. That the coefficient a_2 contains an extra contribution from the vertices was already known to Kac [3], although the precise simple expression here was obtained by Dan Ray (this is referenced by Kac and also later by Cheeger [2], but apparently Ray did not publish his result). A particularly transparent derivation of this corner term appears in a paper by van den Berg and Srisatkunarajah [1].

The heat trace anomaly in the title of our paper is the discrepancy between the heat coefficients in the smooth and polgonal settings. More specifically, it refers to the fact that at least one heat invariant is not continuous with respect to Lipschitz convergence of domains. To phrase this more precisely, let Ω_ϵ be a family of surfaces with *smooth* boundary which converge to a piecewise smoothly bounded domain Ω_0 as $\epsilon \to 0$. We think of Ω_ϵ as Ω_0 with each corner 'rounded out' slightly, but will give a precise formulation in the next paragraph. Denoting the heat trace coefficients for Ω_ϵ by $a_j(\epsilon)$, it will be clear from this definition that

$$\lim_{\epsilon\to 0} a_2(\epsilon) = \lim_{\epsilon\to 0} \frac{1}{12\pi}\left(\int_{\Omega_\epsilon} K_\epsilon\, dA_\epsilon + \int_{\partial\Omega_\epsilon} \kappa_\epsilon\, ds\right) \longrightarrow \frac{1}{12\pi}\left(\int_{\Omega_0} K_0\, dA_0 + \sum_{i=1}^{k}\int_{\gamma_i} \kappa_0\, ds + \sum_{i=1}^{k}\alpha_i\right),$$

where K_ϵ and κ_ϵ are the Gauss curvatures of g_ϵ and the geodesic curvatures of $\partial\Omega_\epsilon$ for every $\epsilon \geq 0$, respectively. The anomaly is simply that this formula does not agree with the expression (1.4). The goal of this paper is to provide a simple explanation for the disagreement between these two expressions.

We now explain the desingularization more precisely. For simplicity, suppose that Ω_0 and Ω_ϵ all lie in some slightly larger ambient open surface $\widetilde{\Omega}$, and that the metrics g_ϵ on Ω_ϵ are all extended to metrics (still denoted g_ϵ) on this larger domain. We assume that this family of metrics converges smoothly on $\widetilde{\Omega}$. Let p be a vertex of Ω_0 and consider the portion of Ω_ϵ in some ball of fixed size around p, $B_c(p)\cap\Omega_\epsilon$. Our main assumption is that the family of pointed spaces $(B_c(p)\cap\Omega_\epsilon, \epsilon^{-2}g_\epsilon, p)$ converges in pointed Gromov-Hausdorff norm, and smoothly, to a noncompact region $Z \subset \mathbb{R}^2$ with smooth boundary, such that at infinity, ∂Z is asymptotic to a cone with vertex at 0 and with opening angle α, the same angle as at the vertex p in (Ω_0, g). This is a slightly different usage of pointed Gromov-Hausdorff convergence since the base point p does not necessarily lie in Ω_ϵ; we can think of this, however, as pointed Gromov-Hausdorff convergence for $(\widetilde{\Omega}, g_\epsilon, p)$.

Note that this definition implies that the distance between p and $\partial\Omega_\epsilon$ is bounded above by a constant times ϵ, and that g_ϵ is a small perturbation, which decreases with ϵ, of the rescaling of the standard flat metric on $Z \cap B_{c/\epsilon}$. For convenience we assume in the rest of this paper that the constant c equals 1. Thus the basic assumption is the existence of a smoothly bounded asymptotically conic region Z in the plane such that $\epsilon^{-1}(\Omega_\epsilon \cap B_1(p))$ converges to Z.

This definition is a very special case of a more general desingularization construction explored carefully in [8] and [9] for the case of degeneration to spaces with isolated conic singularities, and in greater generality in [5]. The goal in these first two papers, as here, is to analyze the behaviour of the heat kernel under this degeneration process. That analysis is quite involved, although it yields much sharper results than can be obtained by the present more naive methods. However, one motivation for the present paper is to show how some very simple rescaling arguments, which are only slight generalizations of ones used (in substantially more sophisticated ways) by Cheeger [2], already yield some interesting results.

Now consider the function

$$G(t,\epsilon) = \operatorname{Tr} H^{\Omega_\epsilon} = \int_{\Omega_\epsilon} H^{\Omega_\epsilon}(t,z,z)\,dz, \tag{1.5}$$

which is smooth on the interior of the quadrant $Q = \{t \geq 0, \epsilon_0 > \epsilon \geq 0\}$; our main theorem concerns its precise regularity at the corner $t = \epsilon = 0$. This will be decribed in terms of its regularity on the parabolic blowup of Q which we denote Q_0. This space is diffeomorphic to Q away from the origin, but has an extra 'front face' F replacing the point $(0,0)$ which encodes all the directions of approach to this point along parabolic trajectories. It is described more carefully in §2 below. One of the goals of this paper, in fact, is to advertise the utility and naturality of this blowup construction.

Theorem 1.6. *Let $(\Omega_\epsilon, g_\epsilon)$ be a family of smooth surfaces with Riemannian metrics which converge in the manner described above to a surface with piecewise smooth boundary (Ω_0, g_0). Then the function $G(t,\epsilon)$ lifts to Q_0 to be polyhomogeneous conormal at all boundary faces and corners.*

Recall that polyhomogeneity means simply that the lift of G has asymptotic expansions at all boundary faces and product type expansions at all corners. The existence of such expansions somehow normalizes our problem. Indeed, the heat trace anomaly is simply the fact that the limit as $\epsilon \searrow 0$ of the second asymptotic coefficient $a_2(\epsilon)$ in the expansion as $t \searrow 0$ is not the same as the second asymptotic coefficient of the heat expansion for Ω_0. The front face F of Q_0 separates where these limits are taken ($t \to 0$ then $\epsilon \to 0$ vs. the other way around), and this extra space allows for the existence of a function which interpolates between these two values. Our second main result describes this function.

Theorem 1.7. *There is a function $C_2(\tau)$ defined along the front face of Q_0, which is smooth in the rescaled time variable $\tau = t/\epsilon^2$, and satisfies*

$$\lim_{\tau \searrow 0} C_2(\tau) = \frac{\chi(\Omega_0)}{6}, \qquad \textit{and}$$

$$\lim_{\tau \nearrow \infty} C_2(\tau) = \frac{\chi(\Omega_0)}{6} + \sum_{j=1}^{k} \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j} - \frac{1}{12\pi}\sum_{j=1}^{k} \alpha_j.$$

Its explicit form includes the finite part of a divergent expansion:

$$C_2(\tau) = \frac{\chi(\Omega_0)}{6} + \sum_{j=1}^{k} \operatorname*{f.p.}_{\epsilon=0} \int_{\{z\in Z_j : |z|<1/\epsilon\}} H^{Z_j}(\tau, z, z)\, dz - \frac{1}{12\pi}\sum_{j=1}^{k} \alpha_j,$$

where Z_j models the collapse at the j^{th} corner.

Remark 1.8. *When Ω_0 is a triangle (or indeed, any simply connected polygon in the plane), then $\chi(\Omega_0) = 1$, so the first and third terms in the formula for $\lim_{\tau\to\infty} C_2(\tau)$ cancel, and we obtain Ray's original formula*

$$\lim_{\tau\to\infty} C_2(\tau) = a_2(0) = \sum_{j=1}^{k} \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j}.$$

This interpolating function $C_2(\tau)$ therefore 'explains' the heat trace anomaly, or alternately, the anomaly is caused by the renormalized heat trace on the complete space Z_j. We also discuss some of the other coefficients in the asymptotic expansions for the lift of G at the various boundary faces and corners of Q_0.

This paper is organized as follows. In §2, we recall some preliminary facts about parabolic blowups and scaling properties of heat kernels and the standard parametrix construction for heat kernels. The proofs of the two theorems are then presented in §3.

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2 Preliminaries

In this section we collect the requisite facts and tools: the behaviour of the heat kernel under scaling of the underlying space, a review of parabolic blowups and polyhomogeneity, and a slight modification of the standard parametrix construction for heat kernels.

2.1 Heat kernels and dilations

The heat kernel transforms naturally under dilations of the domain, or equivalently, of the metric. Let (M, g) be any complete Riemannian manifold with smooth (or piecewise smooth) boundary, and denote by $H^M(t, z, z')$ the minimal heat kernel for the Laplacian with Dirichlet boundary conditions on M.

This is a smooth function on the interior of $\mathbb{R}^+ \times M \times M$ with well-known regularity properties at the various boundaries and corners.

We seek to relate this heat kernel with the one for the same manifold M but with rescaled metric $g_\lambda = \lambda^2 g$, $\lambda \in \mathbb{R}^+$. This will be applied when $M \subset \mathbb{R}^2$, g is the induced Euclidean metric, and we relate its heat kernel to the one for λM, the image of M under the dilation $D_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $z \mapsto \lambda z$. The pullback of the Euclidean metric from λM to M is simply $\lambda^2 g$.

Proposition 2.1. *The heat kernels on M and λM are related by the formula*

$$H^{\lambda M}(\lambda^2 t, \lambda z, \lambda z')\lambda^2 = H^M(t, z, z').$$

Implicit in this formula, we are parametrizing points in λM with points in M via D_λ. To prove this, observe that the heat operator $\partial_t - \Delta_z$ on M transforms homogeneously with respect to the parabolic dilation $(t, z) \mapsto (\lambda^2 t, \lambda z)$. Hence, the expression on the left satisfies the heat equation; the additional λ^2 is the Jacobian factor accounting for the fact that $H^{\lambda M}(0, w, w') = \delta(w - w')$ is homogeneous of order -2 in two dimensions.

2.2 Parabolic blowup

The parabolic dilation $D_\lambda(t, \epsilon) = (\lambda^2 t, \lambda\epsilon)$ motivates the introduction of parabolic blowup Q_0 of the quadrant $Q := [0, \infty)_t \times [0, \epsilon_0)_\epsilon$ at $(0, 0)$. This space is defined as follows. As a set, Q_0 is the disjoint union of $Q \setminus \{(0, 0)\}$ and the orbit space $F = Q \setminus \{(0, 0)\}/\sim$, where $(t, \epsilon) \sim (t', \epsilon')$ if $(t', \epsilon') = D_\lambda(t, \epsilon)$ for some $\lambda > 0$. More concretely, F is diffeomorphic to a closed quarter-circle; it is also identified with the set of all equivalence classes of parametrized curves $\gamma(s) = (t(s), \epsilon(s))$ with $\lim_{s \searrow 0} \gamma(s) = (0, 0)$, $\epsilon(s) = \mathcal{O}(t(s)^2)$, and where

$$\gamma \sim \tilde{\gamma} \Longleftrightarrow \lim_{s \to 0} \frac{\epsilon(s)^2}{t(s)} \Big/ \frac{\tilde{\epsilon}(s)^2}{\tilde{t}(s)} = 1.$$

The curves $t = \tau\epsilon^2$ (parametrized by $s \mapsto (\tau s^2, s)$), $\tau \geq 0$, provide representatives of each equivalence class except the one represented by the t axis. There is a unique minimal $\mathcal{C}^\infty$ structure on Q_0 for which the lifts of smooth functions from Q and the parabolic polar coordinates $r = \sqrt{t + \epsilon^2}$, t/r^2 and ϵ^2/r^2 are all smooth. We label the faces of Q_0 as follows: F is the new front face, and L and R are the left and right side faces (the lifts of $t = 0$

and $\epsilon = 0$, respectively). There is a smooth 'blowdown' map $\beta : Q_0 \longrightarrow Q$ defined in the obvious way.

It is usually more convenient to use projective rather than polar coordinates. There are two such systems,

$$(\tau, \epsilon), \quad \tau = t/\epsilon^2, \qquad \text{and} \quad (t, \eta), \quad \eta = \epsilon/\sqrt{t},$$

which are valid away from R and L, respectively. Thus, for example, τ is an 'angular' variable which vanishes on L, and in this coordinate system $F = \{\epsilon = 0\}$.

Parabolic blowups are described in detail and much greater generality in [6].

2.3 Polyhomogeneous conormal functions

Let M be a manifold with corners. A class of functions which is the natural replacement for (or at least just as good as) the class of smooth functions is the class of polyhomogeneous conormal functions. We refer to [4] for a detailed exposition, but review a few facts about these here.

First recall the space $\mathcal{V}_b$ of all smooth vector fields on M which are tangent to all boundaries of M. If $H_1, \ldots, H_k$ are boundary hypersurfaces of M meeting at a corner of codimension k, with boundary defining functions $x_1, \ldots, x_k$, respectively, and local coordinates $y = (y_1, \ldots, y_{n-k})$ on the corner, then $\mathcal{V}_b$ is spanned over $\mathcal{C}^\infty(M)$ locally near this corner by $\{x_1\partial_{x_1}, \ldots, x_k\partial_{x_k}, \partial_{y_1}, \ldots, \partial_{y_{n-k}}\}$.

A function (or distribution) u is said to be conormal if it has stable regularity with respect to $\mathcal{V}_b$. In other words, there exists a k-tuple of real numbers $\mu_1, \ldots, \mu_k$ so that

$$V_1 \ldots V_\ell \, u \in x_1^{\mu_1} \ldots x_k^{\mu_k} L^\infty(M), \qquad \forall\, \ell \quad \text{and} \; \forall\, V_j \in \mathcal{V}_b.$$

(In particular, the μ_i are independent of ℓ and the V_j.) Examples include monomials $x_1^{s_1} \ldots x_k^{s_k}$ for $s_j \in \mathbb{C}$, as well as products of arbitrary powers of $|\log x_j|$. The special subclass with which we are interested consists of the functions with asymptotic expansions in terms of powers of the boundary defining functions and nonnegative integer powers of the logs of these defining functions, with coefficients which are smooth in all other variables. The expansions are formalized using the notion of an index set I. This consists of a countable sequence of pairs $(\alpha, N) \in \mathbb{C} \times \{\mathbb{N} \cup \{0\}\}$ such that for each $A \in \mathbb{R}$, $\operatorname{Re}\alpha > A$ for all but a finite number of these pairs. Now, the conormal function u has a polyhomogeneous expansion near a corner of codimension k if there are k index sets $I_1, \ldots, I_k$ so that

$$u \sim \sum_{(\alpha_j, N_j) \in I_j} \sum_{\ell_j \leq N_j} x_1^{\alpha_1} (\log x_1)^{\ell_1} \ldots x_k^{\alpha_k} (\log x_k)^{\ell_k} a_{\alpha,\ell}(y),$$

where each coefficient function $a_{\alpha,\ell}$ is $\mathcal{C}^\infty$. Note that since u is already assumed to be conormal, this expansion may be differentiated.

The polyhomogeneous functions on Q and Q_0 with which we shall be concerned are quite simple. None of them will have log terms in their expansions, and the exponents are (not necessarily nonnegative) integers. Thus, for example, near L a polyhomogeneous function u will have expansion in powers of t with coefficients smooth in ϵ; near F in terms of either of the projective coordinate systems, it has an expansion in powers of ϵ with coefficients smooth in τ, or equivalently, in powers of t with coefficients smooth in η; near the corner $L \cap F$ it will have an expansion in powers of τ and ϵ, with coefficients now simply numbers.

The final point to describe here is that if u is polyhomogeneous conormal on Q, then its lift $\beta^* u$ to Q_0 is also polyhomogeneous conormal, and

$$u \sim \sum a_{jk} t^j \epsilon^k \Longrightarrow \beta^* u \sim \sum a_{jk} (\tau \epsilon^2)^j \epsilon^k = \sum a_{jk} \tau^j \epsilon^{2j+k}.$$

(On the other hand, if w is polyhomogeneous on Q_0, then its pushforward to Q is always conormal, but rarely polyhomogeneous.)

2.4 Parametrix construction

We conclude this section by reviewing a parametrix construction for the heat kernel, which is useful because it accurately captures the asymptotics of the true heat kernel as $t \searrow 0$. The construction here is slightly nonstandard, but is well suited for our calculations below.

Let M be a complete Riemannian manifold, possibly with boundary, and suppose that $M = M_1 \cup M_2$ where M_1 and M_2 are two manifolds with boundary with $M_1 \cap M_2 = \Sigma$ a hypersurface. If M has boundary, assume that Σ intersects ∂M transversely, and M_1 and M_2 are manifolds with corners of codimension two. Suppose further that M_j lies in a slightly larger complete manifold M_j', again possibly with boundary, such that for some neighbourhood $\mathcal{U}$ of Σ, $M_j' \cap \mathcal{U} = M \cap \mathcal{U}$.

Taking the heat kernels on each M_j' as given, define

$$\tilde{H}^M(t, z, z') = \sum_{j=1}^{2} \chi_j(z) H^{M_j'}(t, z, z') \chi_j(z'),$$

where χ_j is the characteristic function of M_j in M. In the more customary parametrix construction, the M_j are relatively open in M, and $M_1 \cap M_2$ is also open; the $H^{M_j'}$ are pasted together using cutoff functions $\{\psi_j\}$ and $\{\tilde{\psi}_j\}$ with $\psi_1 + \psi_2 = 1$, where $\operatorname{supp} \psi_j \subset \{\tilde{\psi}_j = 1\}$, and $\operatorname{supp} \tilde{\psi}_j \subset M_j'$. We are using sharp (discontinuous) cutoffs rather than smooth ones, however, so that we can identify certain asymptotic coefficients in the calculations to follow.

Lemma 2.2. *Let $H^M(t, z, z')$ denote the true heat kernel on M, and set*

$$K(t, z) = \tilde{H}^M(t, z, z) - H^M(t, z, z).$$

Then $K(t, z) = \mathcal{O}(t^\infty)$ as $t \searrow 0$.

Proof. Rewrite

$$\begin{aligned}\tilde{H}^M(t, z, z) = \chi_1(z) \left(H^{M_1'}(t, z, z) - H^M(t, z, z) \right) \\ + \chi_2(z) \left(H^{M_2'}(t, z, z) - H^M(t, z, z) \right) + H^M(t, z, z).\end{aligned}$$

By assumption, M_j' agrees with M in a neighbourhood of M_j, so that $H^{M_j'}(t, z, z) - H^M(t, z, z) = \mathcal{O}(t^\infty)$ on the support of χ_j (remember that the small t expansions of these operators are local), and this proves the claim. $\square$

3 Proofs of main theorems

We have now assembled all the requisite facts and can proceed with the proofs of the main theorems.

As in the introduction, let $G(t, \epsilon) = \operatorname{Tr} H^{\Omega_\epsilon}$. If $\beta : Q_0 \to Q$ is the blowdown map, then let $\mathcal{G} = \beta^* G$. We need to analyze the behaviour of $\mathcal{G}$ near each of the faces and corners of Q_0, and for that we shall use the coordinates (τ, ϵ) introduced in §2.2.

We shall make a simplifying assumption about the geometry in order to elucidate the proof. For each i, let S_{α_i} denote the sector in $\mathbb{R}^2$ with opening angle α_i. Choose a smoothly bounded region Z_i in the plane which coincides with S_{α_i} outside $B_{1/2}(0)$, and let $Z_i^\epsilon = B_{1/\epsilon}(0) \cap Z_i$. Then we assume that near each vertex p_i, the restriction of the metric g_ϵ to $B_1(p_i) \cap \Omega_\epsilon$ is isometric to the dilation by the factor ϵ of the region Z_i^ϵ, which obviously lies in the unit ball. The result remains true in the generality with which it was stated earlier, but the proof requires a few more technical steps which are both standard and not particularly germane to the main ideas here. Furthermore, for notational convenience only, we assume that there is only a single vertex p and denote the corresponding smooth model region and sector by Z and S, respectively.

Proof of Theorem 1.6: We first construct a particular family of parametrices for the heat kernel on Ω_ϵ. For any $0 \leq \epsilon < \epsilon_0$, decompose

$$\Omega_\epsilon = \Omega_{\epsilon,1} \cup \Omega'$$

where $\Omega_{\epsilon,1} = \Omega_\epsilon \cap B_1(p)$, and $\Omega' = \Omega_\epsilon \setminus (\Omega_\epsilon \cap B_1(p))$. Note that Ω' is independent of ϵ. Lemma 2.2 shows that

$$H^{\Omega_\epsilon}(t,z,z) = \chi_1(z) H^{\epsilon Z}(t,z,z) + \chi_2(z) H^{\Omega_0}(t,z,z) + K(t,z), \tag{3.1}$$

where χ_1 is the characteristic function of $|z| \leq 1$, $\chi_2 = 1 - \chi_1$, and K is the error term from Lemma 2.2, hence

$$G(t,\epsilon) = \int_{|z|\leq 1} H^{\epsilon Z}(t,z,z)\,dz + \int_{\Omega'} H^{\Omega_0}(t,z,z)\,dz + \int_{\Omega_\epsilon} K(t,z,z)\,dz.$$

We denote the sum on the right side by $\mathrm{I} + \mathrm{II} + \mathrm{III}$, and analyze the lifts of these terms successively.

By Proposition 2.1, $H^{\epsilon Z}(t,z,z') = \epsilon^{-2} H^Z(t/\epsilon^2, z/\epsilon, z'/\epsilon)$, so setting $z = z' = \epsilon w$, we see that

$$\beta^* \mathrm{I} = \int_{|w|\leq 1/\epsilon} H^Z(\tau,w,w)\,dw.$$

This will be the principal term, and we defer its analysis for the moment.

Next, II is independent of ϵ, and it is polyhomogeneous as $t \searrow 0$, with expansion given by integrating the standard heat coefficients $a_j(z)$ over this restricted domain. Hence its lift to Q_0 is clearly polyhomogeneous.

Finally, by Lemma 2.2, III depends on ϵ but decays rapidly in t uniformly in ϵ.

We now examine $\beta^*\mathrm{I}$ more closely. Choose a smoothly bounded compact region W which agrees with Z in $|w| \leq 2$, so that $Z = (W \cap B_1) \cup (S \setminus B_1)$. Using Lemma 2.2 again, write

$$H^Z(t,z,z) = \chi_1(z) H^W(t,z,z) + \chi_2(z) H^S(t,z,z) + K_1(t,z) \tag{3.2}$$

where K_1 is the corresponding error term. Then

$$\beta^*\mathrm{I} = \int_{|w|\leq 1} H^W(\tau,w,w)\,dw + \int_{1\leq |w| \leq 1/\epsilon} H^S(\tau,w,w)\,dw + \int_{|w|\leq 1/\epsilon} K_1(\tau,w,w)\,dw,$$

which we write as $\mathrm{I}' + \mathrm{II}' + \mathrm{III}'$.

We first prove polyhomogeneity of these terms away from the right face R of Q_0. The term I' has an expansion as $\tau \searrow 0$ and is independent of ϵ, so $\beta^*\mathrm{I}'$ is certainly polyhomogeneous in this region. By Lemma 2.2 again, K_1 decreases rapidly as $\tau \to 0$, so this term is also polyhomogeneous there. Note too that by the explicit form of the error term in the proof of that lemma, and using the dilation properties of H^Z and H^S again, $K_1(\tau, z) = \mathcal{O}(|z|^{-\infty})$ uniformly for τ in any bounded set, so its integral over $|z| \leq 1/\epsilon$ is also bounded independently of ϵ.

To analyze the remaining term, set

$$D(R) := \int_{|w| \leq R} H^S(1, w, w)\, dw.$$

By Proposition 2.1, $\mathrm{II}'(\epsilon, \tau) = D(1/\epsilon\sqrt{\tau}) - D(1/\sqrt{\tau})$, so it will suffice to show that D has an expansion in powers of $1/R$ as $R \to \infty$. For this, we appeal to a calculation by van den Berg and Srisatkunarajah [1], who prove that

$$D(R) = \frac{\alpha R^2}{8\pi} - \frac{R^2}{2\pi} \int_0^1 e^{-R^2 y^2} \sqrt{1 - y^2}\, dy + \frac{\pi^2 - \alpha^2}{24\pi\alpha} + \mathcal{O}(e^{-cR^2}) \qquad (3.3)$$

for some $c > 0$ independent of R. Only the polyhomogeneous structure of the second term on the right is nonobvious. For that, we may as well replace the upper limit of integration by $1/2$ since the integral from $1/2$ to 1 decreases exponentially in R. Using the Taylor series for $\sqrt{1 - y^2}$ at $y = 0$, we find that

$$\frac{R^2}{2\pi} \int_0^{1/2} e^{-R^2 y^2} (1 - \frac{1}{2} y^2 - \frac{1}{4} y^4 - \ldots)\, dy \sim \frac{R}{4\sqrt{\pi}} - \frac{1}{16\sqrt{\pi} R} + \mathcal{O}(R^{-3}),$$

and this completes the proof of polyhomogeneity of $\beta^*\mathrm{I}$ for τ in any bounded set.

To finish the proof, we must analyze the behaviour of $\beta^*\mathrm{I}$ as $\tau \nearrow \infty$. Switch to the coordinates t, η, so $\epsilon = \eta\sqrt{t}$, and $\tau = \eta^{-2}$. It is now more convenient to use the standard representation of the heat kernel in terms of the resolvent:

$$H^Z = \int_\Gamma e^{-\tau\lambda} R_Z(\lambda)\, d\lambda, \qquad \text{where} \quad R_Z(\lambda) = (-\Delta_Z - \lambda)^{-1}. \qquad (3.4)$$

Here Γ is a path surrounding the spectrum of Δ_Z, for example, the two half-lines $\mathrm{Im}\,\lambda = \pm(\alpha \mathrm{Re}\,\lambda + \beta)$, $\alpha, \beta > 0$, joined by the half-circle $|\lambda| = \beta$,

$\operatorname{Re}\lambda \leq 0$, traversed in the counterclockwise direction. In general it is a subtle matter to deduce the fact that H^Z has an expansion in powers of $1/\tau$ at large times since this depends on the fine structure of the resolvent near the threshold $\lambda = 0$. However, in this case we already have sufficient information about the heat kernel on S that this is not hard. Choose a partition of unity $\{\psi_1, \psi_2\}$ on Z such that $\psi_1 = 1$ in $|z| \leq 3/4$ and $\psi_2 = 1$ in $|z| \geq 5/4$, and that both $Z \cap W$ and $Z \cap S$ contain the region $Z \cap \{3/4 \leq |z| \leq 5/4\}$. Choose other cutoff functions $\tilde{\psi}_j$ such that $\tilde{\psi}_j = 1$ on $\operatorname{supp}\psi_j$. Let R_W and R_S denote the resolvents for Δ_W and Δ_S, with Dirichlet boundary conditions, and define the parametrix

$$\tilde{R}_Z(\lambda) = \tilde{\psi}_1 R_W(\lambda)\psi_1 + \tilde{\psi}_2 R_S(\lambda)\psi_2.$$

Then

$$(\Delta_Z - \lambda)\tilde{R}_Z(\lambda) = I + [\Delta_Z, \tilde{\psi}_1]R_W(\lambda)\psi_1 + [\Delta_Z, \tilde{\psi}_2]R_S(\lambda)\psi_2 := I + E(\lambda).$$

Since the singular supports of both R_W and R_S are on the diagonal, and the support of $[\Delta, \tilde{\psi}_j]$ is disjoint from that of ψ_j, we see that $E(\lambda)$ is a holomorphic family of operators (for $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$) which maps $L^2(Z)$ into $\mathcal{C}_0^\infty(Z)$. Also, since $\Delta_Z - \lambda$ is invertible for λ in this region, $I + E(\lambda)$ is also invertible there. We write its inverse as $I + F(\lambda)$, so that

$$R_Z(\lambda) = \tilde{R}_Z(\lambda) + \tilde{R}_Z(\lambda)F(\lambda). \tag{3.5}$$

The relationships $(I + E(\lambda))(I + F(\lambda)) = (I + F(\lambda))(I + E(\lambda)) = I$ imply that

$$F(\lambda) = -E(\lambda) + E^2(\lambda) + E(\lambda)F(\lambda)E(\lambda),$$

hence $F(\lambda)$ is also smoothing and maps $L^2(Z)$ into $\mathcal{C}_0^\infty(Z)$; the second term on the right in (3.5) has the same mapping properties.

Finally, the form of the expansion of $F(\lambda)$ for λ near 0 (away from the positive real axis) is the precisely the same as that of $E(\lambda)$, which in turn is the same as that of $\tilde{R}_Z(\lambda)$, and hence finally as that of $R_S(\lambda)$.

The decay of each of term as $|\lambda| \to \infty$ is straightforward, so we can write

$$\begin{aligned}
\int_{|z|\leq 1/\eta\sqrt{t}} H^Z(\eta^{-2}, z, z)\, dz &= \int_{|z|\leq 1/\eta\sqrt{t}} \left(\int_\Gamma e^{-\lambda/\eta^2} R_Z(\lambda)\, d\lambda \right)(z,z)\, dz \\
&= \int_{|z|\leq 2} \left(\int_\Gamma e^{-\lambda/\eta^2} \psi_1 R_W(\lambda)\, d\lambda \right)(z,z)\, dz \\
&+ \int_{1/2\leq|z|\leq 1/\eta\sqrt{t}} \left(\int_\Gamma e^{-\lambda/\eta^2} \psi_2 R_S(\lambda)\, d\lambda \right)(z,z)\, dz \\
&+ \int_{|z|\leq 1/\eta\sqrt{t}} \left(\int_\Gamma e^{-\lambda/\eta^2} \tilde{R}_Z(\lambda) F(\lambda)\, d\lambda \right)(z,z)\, dz.
\end{aligned}$$

The inner integrand in the first term on the right extends holomorphically to a neighbourhood of $\lambda = 0$, so the contour can be moved to lie entirely in the right half-plane, which shows that this term decreases exponentially in $1/\eta$. The second term is polyhomogeneous by the explicit analysis of the function $D(R)$ above. The fact that the final term has an expansion follows from the existence of asymptotics of $F(\lambda)$ for λ near 0. This completes the proof of the polyhomogeneity of $\mathcal{G}$ on Q_0. □

Proof of Theorem (1.7) This consists of examining the terms in the expansion of $\mathcal{G}$ at the various boundary faces.

First, at L, away from F we may use the variables (t, ϵ), and

$$G(t,\epsilon) \sim \sum_{j=0}^{\infty} a_j(\epsilon) t^{-1+j/2}.$$

Near L ∩ F, we substitute $t = \epsilon^2 \tau$ to get

$$\mathcal{G}(\tau,\epsilon) \sim \sum_{j=0}^{\infty} a_j(\epsilon) \tau^{-1+j/2} \epsilon^{-2+j}. \tag{3.6}$$

The coefficients $a_j(\epsilon)$ are polyhomogeneous as $\epsilon \to 0$ by Theorem 1.6.

At R, away from $t = 0$,

$$G(t,\epsilon) \sim \sum_{j=0}^{\infty} B_j(t) \epsilon^j;$$

here $B_0(t) = \operatorname{Tr} H^{\Omega_0}$. Near F ∩ R we use the coordinates t and $\eta = \epsilon/\sqrt{t}$ to compute

$$\mathcal{G}(t,\eta) \sim \sum_{j=0}^{\infty} B_j(t) \eta^j t^{j/2}.$$

Again, the coefficients $B_j(t)$ are polyhomogeneous in t.

Finally, near F, we use the coordinates (τ, ϵ), so the expansion is in powers of ϵ, and by (3.6) it is

$$\mathcal{G}(\tau, \epsilon) \sim \sum_{j=0}^{\infty} C_j(\tau)\epsilon^{-2+j}.$$

We shall identify the coefficients C_0, C_1 and C_2.

By our analysis of the terms I′, II′, III′, II and III, we see that only II′ and II contribute to the coefficients of ϵ^{-2} and ϵ^{-1}. Substituting directly from the expansions of these two terms (using the McKean-Singer asymptotics on Ω' for II and the first terms in the expansion of $D(1/\epsilon\sqrt{\tau})$ for II′), and then using the definition of the finite part at $\epsilon = 0$ of I, we have

$$\mathcal{G}(\tau, \epsilon) \sim \frac{1}{\epsilon^2 \tau}\left(\frac{|\Omega'|}{4\pi} + \frac{\alpha}{8\pi}\right) - \frac{1}{\epsilon \tau^{1/2}}\left(\frac{|\partial\Omega'|}{8\sqrt{\pi}} + \frac{1}{4\sqrt{\pi}}\right)$$
$$+ \frac{1}{12\pi}\left(\int_{\Omega'} K\, dA + \int_{\partial\Omega'} \kappa\, ds\right) + \underset{\epsilon=0}{\text{f.p.}} \int_Z H^Z(\tau, w, w)\, dw + \mathcal{O}(\epsilon).$$

In other words,

$$C_0(\tau) = \frac{1}{\tau}\left(\frac{|\Omega'|}{4\pi} + \frac{\alpha}{8\pi}\right),$$
$$C_1(\tau) = -\frac{1}{\sqrt{\tau}}\left(\frac{|\partial\Omega'|}{8\sqrt{\pi}} + \frac{1}{4\sqrt{\pi}}\right),$$

and

$$C_2(\tau) = \frac{1}{12\pi}\left(\int_{\Omega'} K dA + \int_{\partial\Omega'} \kappa ds\right) + \underset{\epsilon=0}{\text{f.p.}} \int_Z H^Z(\tau, w, w)\, dw.$$

This simplifies using the following observations: first, calculating the area of a circular sector of opening α gives $|\Omega_0 \cap B_1| = \alpha/2$, so the coefficient of $\epsilon^{-2}\tau^{-1}$ is just $|\Omega_0|$; second, the sides of this circular sector are straight lines, so $|\partial\Omega_0 \cap B_1| = 2$, which means that the next coefficient is $-|\partial\Omega_0|/8\sqrt{\pi}$; finally, since g_0 is flat in $\Omega_0 \cap B_1$, $K \equiv 0$ there, so using that the contribution from 'turning the corner' at p in the boundary integral is α, we find that

$$\int_{\Omega'} K\, dA + \int_{\partial\Omega'} \kappa\, ds = 2\pi\chi(\Omega_0) - \alpha.$$

This means that the first part of the coefficient of ϵ^0 reduces to $\chi(\Omega_0)/6 - \alpha/12\pi$. In other words,

$$C_2(\tau) = \underset{\epsilon=0}{\text{f.p.}} \int_Z H^Z(\tau, w, w)\, dw + \frac{1}{6}\chi(\Omega_0) - \frac{\alpha}{12\pi}. \tag{3.7}$$

We conclude by calculating its behaviour for small and large τ. Using the small τ asymptotics, we see that

$$\int_{|w|<1/\epsilon} H^Z(\tau, w, w)\, dw \sim \frac{|Z \cap B_{1/\epsilon}|}{4\pi}\tau^{-1} - \frac{|\partial Z \cap B_{1/\epsilon}|}{8\sqrt{\pi}}\tau^{-1/2} + \frac{1}{12\pi}\int_{\partial Z} \kappa\, ds + \mathcal{O}(\epsilon\tau^{1/2}),$$

which means that the finite part is equal to $\alpha/12\pi$, so the limit of C_2 as $\tau \to 0$ is $\chi(\Omega_0)/6$, as claimed.

Finally, we use the dilation one more time to calculate that

$$\int_{|w|\leq 1/\epsilon} H^Z(\tau, w, w)\, dw = \int_{|w|\leq 1/\epsilon\sqrt{\tau}} H^{Z/\sqrt{\tau}}(1, w, w)\, dw.$$

Noting that $\epsilon\sqrt{\tau} = \sqrt{t}$, and since $Z/\sqrt{\tau}$ converges to the sector S as $\tau \to \infty$, we can use the expansion (3.3) to see that the finite part is indeed $(\pi^2 - \alpha^2)/24\pi\alpha$. Therefore, in general, with an arbitrary number of vertices,

$$\lim_{\tau\to\infty} C_2(\tau) = \frac{\chi(\Omega_0)}{6} + \sum_{j=1}^{k} \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j} - \frac{1}{12\pi}\sum_{j=1}^{k} \alpha_j;$$

in particular, if Ω_0 is a triangle, its Euler characteristic is 1, so the first and third terms cancel.

This completes the proof. □

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