Spectral Geometry and Asymptotically Conic Convergence

Julie Marie Rowlett *

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Abstract

In this paper we define *asymptotically conic convergence* in which a family of smooth Riemannian metrics degenerates to have an isolated conic singularity. For a conic metric (M_0, g_0) and an asymptotically conic (scattering) metric (Z, g_z) we define a non-standard blowup, the *resolution blowup,* in which the conic singularity in M_0 is resolved by Z. Equivalently, the resolution blowup resolves the boundary of the scattering metric using the conic metric; the resolution space is a smooth compact manifold. This blowup induces a smooth family of metrics $\{g_\epsilon\}$ on the compact resolution space M, and we say (M, g_ϵ) converges *asymptotically conically* to (M_0, g_0) as $\epsilon \to 0$.

Let Δ_ϵ and Δ_0 be geometric Laplacians on (M, g_ϵ) and (M_0, g_0), respectively. Our first result is convergence of the spectrum of Δ_ϵ to the spectrum of Δ_0 as $\epsilon \to 0$. Note that this result implies spectral convergence for the k-form Laplacian under certain geometric hypotheses. This theorem is proven using rescaling arguments, standard elliptic techniques, and the b-calculus of [26]. Our second result is technical: we construct a parameter (ϵ) dependent heat operator calculus which contains, and hence describes precisely, the heat kernel for Δ_ϵ as $\epsilon \to 0$. The consequences of this result include: the existence of a polyhomogeneous asymptotic expansion for H_ϵ as $\epsilon \to 0$, with uniform convergence down to $t = 0$. To prove this result we construct manifolds with corners (heat spaces) using both standard and non-standard blowups, on which we construct corresponding heat operator calculi. A parametrix construction modeled after the heat kernel construction of [26] and a maximum principle type argument complete this proof.

1 Introduction

Cheeger and Colding wrote a series of three papers between 1997 and 2000 on the Gromov-Hausdorff limits of families of smooth, connected Riemannian manifolds with lower Ricci curvature bounds [5], [6], [7]. They proved Fukaya's conjecture of 1987 [10]: on any pointed Gromov-Hausdorff limit space of a family $\{M_i^n\}$ of connected Riemannian manifolds with

*Email: rowlett@math.ucsb.edu. Department of Mathematics, South Hall Room 6607, University of California, Santa Barbara, California 93106. AMS Classification primary: 58J50, secondary: 58J35.

Ricci curvature bounded below, a self adjoint extension of the scalar Laplacian can be defined, with discrete spectrum and Lipschitz eigenfunctions in the compact case, so that the eigenvalues and eigenfunctions of the scalar Laplacians on M_i^n behave continuously as $i \to \infty$. An example, provided by Perelman, showed that the results of Cheeger and Colding do not extend to the k-form Laplacian or to more general geometric Laplacians. In 2002, Ding proved convergence of the heat kernels and Green's functions in the same setting [8]. The estimates are uniform for time bounded strictly away from zero. These results are impressive; the only hypothesis is a lower Ricci curvature bound! It would be useful to prove a more general spectral convergence result for geometric Laplacians and to obtain uniform estimates on the heat kernels for all time. In order to obtain such results, it becomes necessary to impose more structure on the manifolds and the way in which they converge to a singular limit space.

Let M be a fixed compact manifold with Riemannian metric and let H be an embedded orientable hypersurface with defining function x and smooth metric g_H, and let

$$g_\epsilon := dx^2 + (\epsilon^2 + x^2) g_H \quad \epsilon \in [0, 1).$$

As $\epsilon \to 0$, $g_\epsilon \to x^2 g_H + dx^2$, which has an isolated conic singularity at $x = 0$. Geometrically, M is pinched along the hypersurface H as $\epsilon \to 0$, and the resulting metric has a conic singularity as $x \to 0$. The study of this metric collapse is the content of the 1990 thesis of McDonald [25].

In 1995, Mazzeo and Melrose [24] developed pseudodifferential techniques to describe the behavior of the spectral geometry under another specific type of metric collapse known as *analytic surgery.* As $\epsilon \to 0$ the metrics

$$g_\epsilon = \frac{|dx|^2}{x^2 + \epsilon^2} + h \longrightarrow \frac{|dx|^2}{x^2} + h = g_0;$$

g_0 is an exact b-metric on the compact manifold with boundary $\bar{M}$ obtained by cutting M along H and compactifying as a manifold with boundary, hence the name, *analytic surgery.* Under certain assumptions on the associated Dirac operators,[1] [24] proved

$$\lim_{\epsilon \to 0} \eta(\eth_\epsilon) = \eta_b(\eth_{\bar{M}}),$$

where $\eta_b(\eth_{\bar{M}})$ is the b-version of the eta invariant introduced by Melrose. These results were proven by analyzing the resolvent family of the Dirac operators $\eth_\epsilon$ *uniformly* near zero. This led to a precise description of the behaviour of the small eigenvalues. We obtain here similarly uniform results for the heat kernels and expect similar applications.

The convergence we consider, *asymptotically conic (ac) convergence,* is more restrictive than that of [5], [6], [7] and [8], but more general than the conic degeneration of [25]. We note, however, that ac convergence does not require Ricci curvature bounds. The conic

[1] In a later collaboration with Hassel [15] these hypotheses were removed.

collapse of [25], the analogous smooth collapse of a higher codimension submanifold and the collapse of an open neighborhood of the manifold with some restrictions on the local geometry all fit this new definition. Before stating our spectral convergence results for geometric Laplacians, we recall their definition.

Definition 1. *Let (E, ∇) be a Hermitian vector bundle over a Riemannian manifold (M, g) with metric-compatible connection ∇. A geometric Laplacian is an operator Δ acting on sections of E which has the form*

$$\Delta = \nabla^*\nabla + R,$$

where R is a non-negative self-adjoint endomorphism of E. By the Weitzenböck Theorem [28], the Laplacian on k-forms is a geometric Laplacian, as is the Hodge Laplacian and the conformal Laplacian; any geometric Laplace-type operator is a geometric Laplacian.

Our results are the following.

Spectral Convergence: Theorem 1. *Let (M_0, g_0) be a compact Riemannian n-manifold with isolated conic singularity, and let (Z, g_z) be an asymptotically conic space, with $n \geq 3$. Assume (M, g_ϵ) converges asymptotically conically to (M_0, g_0). Let (E_0, ∇_0) and (E_z, ∇_z) be Hermitian vector bundles over (M_0, g_0) and (Z, g_z), respectively, so that each of these bundles in a neighborhood of the boundary is the pullback from a bundle over the cross section (Y, h). Let Δ_0, Δ_z be the corresponding Friedrich's extensions of geometric Laplacians, and let Δ_ϵ be the induced geometric Laplacian on (M, g_ϵ). Assume Δ_z has no $\mathcal{L}^2$ nullspace. Then the accumulation points of the spectrum of Δ_ϵ as $\epsilon \to 0$ are precisely the points of the spectrum of Δ_0, counting multiplicity.*

The setting for our next result is the acc heat space, a manifold with corners constructed in section 7.

Heat Kernel Convergence: Theorem 2. *Let (M_0, g_0) be a compact Riemannian n-manifold with isolated conic singularity, and let (Z, g_z) be an asymptotically conic space, with $n \geq 2$. Assume (M, g_ϵ) converges asymptotically conically to (M_0, g_0). Let (E_0, ∇_0) and (E_z, ∇_z) be Hermitian vector bundles over (M_0, g_0) and (Z, g_z), respectively, so that each of these bundles in a neighborhood of the boundary is the pullback from a bundle over the cross section (Y, h). Let Δ_0, Δ_z be the corresponding Friedrich's extensions of geometric Laplacians, and let Δ_ϵ be the induced geometric Laplacian on (M, g_ϵ). Then the associated heat kernels H_ϵ have a full polyhomogeneous expansion as $\epsilon \to 0$ on the asymptotically conic convergence (acc) heat space with the following leading terms:*

- *At the conic front face, F_{0101}, $H(z, z', t, \epsilon) \to H_0(z, z', t)$, the heat kernel for (M_0, g_0).*
- *At the rescaled b front face, $F_{1010,2}$, $H(z, z', t, \epsilon) \to (\rho_{1010,2}) H_b(\tau)$, the b heat kernel with rescaled time variable τ.*

- *At the exact conic front face,* $F_{1111,2}$*,* $H(z, z', t, \epsilon) \to (\rho_{1111,2})^2 H_0(\tau)$*, the heat kernel for the exact cone with rescaled time variable* τ*.*
- *At the side faces* F_{1001}, F_{0110} *and the residual b face* F_{1010}*, the heat kernel vanishes to infinite order.*[2]

This convergence is uniform in ϵ *for all time and moreover, the error term is bounded by* $C_N \epsilon t^N$ *as* $t \to 0$*, for any* $N \in \mathbb{N}_0$*.*

Remarks:

- This theorem immediately implies the uniform convergence

$$H(z, z', t, \epsilon) \to H_0(z, z', t), \quad T > t > 0,$$

  as well as the convergence

$$H(z, z', t, \epsilon) \to H_0(z, z', t) + O(\epsilon), \quad t, \epsilon \to 0,$$

  with explicit error given by the leading terms above in the polyhomogeneous expansion of $H(z, z', t, \epsilon)$ on the acc heat space as $\epsilon \to 0$.

- We have dropped the half density factor,

$$(\rho_{1111,2})^{n+1} (\rho_{1010,2})^n (\rho_{1010})^{n-2} (\rho_{d2})^{(n+2)/2} (\rho_{1001}\rho_{0110})^{(n-1)/2} \sqrt{\nu},$$

  since it includes extra vanishing factors as a result of the blowups in the acc heat space.

These theorems are proven in sections 5 and 7, respectively. In section 2 we define the resolution blowup and ac convergence. Sections 3 and 4 contain a brief review of geometric and analytic results and terminology on manifolds with corners. In section 6 we construct the heat spaces and heat operator calculi that will be used to prove the main theorem in section 7.

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[2]Above, ρ_{wxyz} is the defining function for boundary face F_{wxyz}.

2 Asymptotically Conic Convergence

The definition of *asymptotically conic convergence* involves three geometries: a family of smooth metrics on a compact manifold, a conic metric on a compact (incomplete) manifold, and an *asymptotically conic* or ac scattering metric on a compact (complete) manifold with boundary.

First, we define *ac* scattering metric.

Definition 2. *An ac scattering metric $(\bar{Z}, g_z)$ is a smooth metric on a compact n-manifold with boundary, $\partial\bar{Z} = (Y, h)$ a smooth compact $(n-1)$ manifold. $\bar{Z}$ has a product decomposition $(0, r_1)_r \times Y$ in a neighborhood of the boundary defined by $r = 0$, so that on this neighborhood,*

$$g_z = \frac{dr^2}{r^4} + \frac{h(r)}{r^2}, \quad h(r) \to h \text{ as } r \to 0,$$

in other words, h extends to a $\mathcal{C}^\infty$ tensor on $[0, r_1)$. Uniquely associated to the ac scattering metric $(\bar{Z}, g_z)$ is the complete non-compact manifold Z with asymptotically conic end; Z is known as an ac space.[3] *Letting $\rho = \frac{1}{r}$, there is a compact subset $K_z \subset Z$ so that*

$$Z - K_z \cong (1/r_1, \infty)_\rho \times Y, \quad g_z|_{Z-K_z} = d\rho^2 + \rho^2 h(1/\rho).$$

We will frequently identify a neighborhood $U \subset \bar{Z}$ which is diffeomorphic to $(0,1)_r \times Y$ with $(0,1)_r \times Y$ and we will identify the metric on U with $\frac{dr^2}{r^4} + \frac{h(r)}{r^2}$. A familiar example of an ac scattering metric is the standard metric on the radial compactification of $\mathbb{R}^n$ with boundary $\mathbb{S}^{n-1}$ at infinity.

Next, we define the conic metric on a compact manifold.

Definition 3. *Let M_c be a compact metric space with Riemannian metric g. Then, (M_c, g) has an isolated conic singularity at the point p and g is called a* conic metric *if the following hold.*

1. *$(M_c - \{p\}, g)$ is a smooth, open manifold.*

2. *There is a neighborhood N of p and a function $x : N - \{p\} \to (0, x_1]$ for some $x_1 > 0$, such that $N - \{p\}$ is diffeomorphic to $(0, x_1]_x \times Y$ with metric $g = dx^2 + x^2 h(x)$ where (Y, h) is a compact, smooth $n - 1$ manifold and $\{h(x)\}$ is a smooth family of metrics on Y converging to h as $x \to 0$, in other words, h extends to a $\mathcal{C}^\infty$ tensor on $[0, x_1)_x \times Y$.*

[3]Note that asymptotically conic spaces are sometimes called "asymptotically locally Euclidean," or *ALE*. However, that term is often used for the more restrictive class of spaces that are asymptotic at infinity to a cone over a quotient of the sphere by a finite group, so to avoid confusion, we use the term *asymptotically conic*.

Associated to a manifold with isolated conic singularity is the manifold with boundary obtained by blowing up the cone point, adding a copy of (Y, h) at this point. Then, x is a boundary defining function for the boundary, (Y, h). We use $M_0^0 = M_c - \{p\}$ to denote the smooth incomplete conic manifold, M_c to denote the metric space closure of M_0^0, and M_0 to denote the associated manifold with boundary. We will frequently identify a neighborhood $U \subset M_0$ which is diffeomorphic to $(0,1)_x \times Y$ with $(0,1)_x \times Y$ and we will identify the metric on U with $dx^2 + x^2 h(x)$.

In the definition of *resolution blowup* we use the notation $M \cup_\phi N$ for a smooth manifold constructed from the smooth manifolds M and N with a diffeomorphism ϕ from $V \subset N$ to $U \subset M$ that gives the equivalence relation, $V \ni p \sim \phi(p) \in U$. $M \cup_\phi N$ is the disjoint union of M and N modulo the equivalence relation of ϕ. The smooth structure on $M \cup_\phi N$ and the topology is induced by that of M and N.

Definition 4. *Let (M_0, g_0) be a compact n manifold with isolated conic singularity and let (Z, g_z) be an asymptotically conic space of dimension n, so that $\partial M_0 = \partial \bar{Z} = (Y, h)$. Then,*

$$M_0^0 = K_0 \cup V_0,$$

where $V_0 \cong (0, x_1)_x \times Y$, and K_0 is compact. With this diffeomorphism

$$g_0 = dx^2 + x^2 \tilde{h}(x) \quad \text{on } (0, x_1) \times Y.$$

We may assume the boundary of K_0 in M_0^0 is of the form $\partial K_0 = \{x = x_1\} \cong Y$, and we may extend x smoothly to K_0 so that $2x_1 > x \geq x_1$ on K_0. Similarly,

$$Z = K_z \cup V_z,$$

where $V_z \cong (\rho_1, \infty)_\rho \times Y$, and K_z is compact. With this diffeomorphism

$$g_z = d\rho^2 + \rho^2 h(\rho, y) \quad \text{on } (\rho_1, \infty) \times Y.$$

We may assume that ∂K_z is of the form $\partial K_z = \{\rho = \rho_1\} \cong Y$, and we similarly extend ρ smoothly to K_z so that $\rho_1 \geq \rho > \rho_1/2$ on K_z.

Let $\delta = \min\{x_1, 1/\rho_1\}$. Then for $0 < \epsilon < \delta$, and $R > \rho_1$, let

$$M_{0,\epsilon} = \{(x, y) \in M_0 : x > \epsilon\}, \quad Z_R = \{(\rho, y) \in Z : \rho < R\}.$$

The **resolution blowup** *of (M_0, g_0) by (Z, g_z) is,*

$$M_\epsilon := M_{0,\epsilon} \cup_{\phi_\epsilon} Z_{1/\epsilon},$$

where the joining map ϕ is defined for each ϵ by

$$\phi_\epsilon : M_{0,\epsilon} - M_{0,\delta} \to Z_{1/\epsilon} - Z_{1/\delta}, \quad \phi_\epsilon(x, y) = \left(\frac{x}{\delta \epsilon}, y \right).$$

For $\delta > \epsilon > \epsilon' > 0$, the manifolds M_ϵ and $M_{\epsilon'}$ are diffeomorphic, and so the resolution blowup of M_0 by Z which we call M is unique up to diffeomorphism.

Remark: The resolution blowup resolves the singularity in a conic manifold using an ac scattering space. Instead of resolving the singularity in M_0 using Z, we may equivalently define the resolution blowup to resolve the boundary of $\bar{Z}$ using M_0.[4]

The ac single space, analogous to the analytic surgery single space in [24], is the setting for the definition of ac convergence.

Definition 5. *Let (M_0, g_0) be a conic metric and let $(\bar{Z}, g_z)$ be a scattering metric; assume both are dimension n with the same cross section (Y, h) at the boundary, and assume $\delta = 1$ (definition 4). Then, $M_0 \cong ((0,1)_x \times Y) \cup K_0$ and $\bar{Z} \cong ((0,1)_r \times Y) \cup K_z$ with $\partial K_0 \cong Y \cong \partial K_z$. The* asymptotically conic convergence (acc) single space $\mathcal{S}$ *is*

$$\mathcal{S} := [0,1)_x \times [0,1)_r \times Y \cup (K_0 \times \{x = 1, r \neq 1\}) \cup (K_z \times \{r = 1, x \neq 1\}).$$

The smooth structure of $\mathcal{S}$ is induced by that of M_0 and $\bar{Z}$. Namely, smooth functions on $\mathcal{S}$ are functions which are smooth jointly in x and r on $(0,1)_x \times (0,1)_r \times Y$, smoothly extend to a smooth function on K_0 at $x = 1$, on K_z at $r = 1$, on M_0 at $r = 0$, and on $\bar{Z}$ at $x = 0$. To give a precise description of the metrics considered here, we define below the acc tensor, a smooth, polyhomogeneous, symmetric 2-cotensor on the acc single space.

Definition 6. *Let $\mathcal{S}$ be the acc single space associated to M_0 and $\bar{Z}$ as in definition 5. Let $\epsilon(p) = x(p)r(p) : \mathcal{S} \to [0,1)$, where x, r are extended to K_0 and K_z respectively to be identically* 1. *We define the* acc tensor $\mathcal{G}$ *as follows:*

$$\mathcal{G} = \begin{cases} \frac{1}{2}\left(dx^2 + x^2\left(h(x) + r^2\left(\frac{dr^2}{r^4} + \frac{h(r)}{r^2}\right)\right)\right) & x, r \in (0,1) \\ (xr)^2 (g_z|_{K_z}) & r = 1 \\ g_0|_{K_0} & x = 1 \end{cases}$$

For $0 < \epsilon < 1$, let $M_\epsilon = \{xr = \epsilon\} \subset \mathcal{S}$; note that this M_ϵ is diffeomorphic to the resolution blowup M of M_0 by Z.[5] *Assume the $h(x)$ has a smooth expansion in x,*

$$h(x) \sim h_0 + xh_1 + x^2 h_2 + \dots$$

which is valid for $x > 0$, where $h_0, h_1, \dots$ are smooth symmetric 2-cotensors on Y. This is to ensure $\mathcal{G}$ is polyhomogeneous at the codimension 2 corner in $\mathcal{S}$. Then, the family of metrics $\{g_\epsilon = \mathcal{G}|_{M_\epsilon}\}$ on M is said to converge asymptotically conically *to (M_0, g_0).*

[4] Let $r = 1/\rho$ be the defining function for $\partial\bar{Z}$. The resolution blowup of $\bar{Z}$ by M_0 is

$$M_0 \cup_{\psi_\epsilon} \bar{Z}, \quad \psi_\epsilon(x, y) = \left(\frac{\epsilon\delta}{x}, y\right),$$

where the patching map ψ_ϵ is defined on $M_0 - K_0$ with image $(0, \delta)_r \times Y$ a neighborhood of $\partial\bar{Z}$. The resulting smooth compact resolution space is diffeomorphic to M.

[5] In the definition of the acc single space we have assumed δ from the definition of resolution blowup is 1. This is to simplify calculations. On $\{xr = \epsilon\}$ we then have $r = \frac{\epsilon}{x}$, equivalently $x = \frac{\epsilon}{r}$, so letting $\delta = 1$ in the definition of resolution blowup the identification of $\{xr = \epsilon\} \subset \mathcal{S}$ and the resolution blowup M_ϵ follows immediately.

Remarks

1. The acc single space has two boundary hypersurfaces at $\epsilon = 0$; these are diffeomorphic to M_0 at $r = 0$ and $\bar{Z}$ at $x = 0$, and they meet in a codimension 2 corner diffeomorphic to Y. There are also boundary hypersurfaces at $\epsilon = 1$ which we ignore since we are interested in $\epsilon \to 0$. With the metric $\mathcal{G}$, $(0,1)_x \times Y \times (0,1)_r$ contains a submanifold diffeomorphic to a cone over $(0,1)_r \times Y \subset \bar{Z}$ with radial variable x.

2. Since g_0 is a smooth metric on $M_0 \cong ((0,1)_x \times Y) \cup K_0$ and g_z is a smooth metric on $\bar{Z} \cong ((0,1)_r \times Y) \cup K_z$, $\mathcal{G}$ is a smooth symmetric 2-cotensor on $\mathcal{S}$ which extends smoothly across neighborhoods where it is piecewise defined and is polyhomogeneous at all boundary faces of $\mathcal{S}$.

3. At $r = 0$, $\mathcal{G}$ restricts to $\mathcal{G}|_{\{r=0\}} = g_0$; as $x \to 0$, $\mathcal{G}$ vanishes to order 2 in the tangential directions.

4. On $M_\epsilon \subset \mathcal{S}$ where $0 < r(p), x(p) < 1$,

$$r = \frac{\epsilon}{x} \qquad \Longrightarrow \; dr^2 = \frac{\epsilon^2}{x^4} dx^2,$$

so

$$g_\epsilon = dx^2 + x^2 h(x) = \epsilon^2 \left(\frac{dr^2}{r^4} + \frac{h(r)}{r^2} \right).$$

5. On M_ϵ when $x \equiv \epsilon$, $r \equiv 1$ and this subset is diffeomorphic to K_z with metric $g_\epsilon = \epsilon^2 g_z|_{K_z}$. Similarly, when $x \equiv 1$, $r \equiv \epsilon$ and this subset is diffeomorphic to K_0 with metric $g_\epsilon = g_0 + O(\epsilon)^2$.

The following lemma is useful for visualizing ac convergence and for proving spectral convergence.

Lemma 1. *Let (M_0, g_0) and (Z, g_z) be as in definitions 4, 5, 6 and let (M, g_ϵ) converge asymptotically conically to (M_0, g_0). Then, there exists a family of diffeomorphisms $\{\phi_\epsilon\}$ from a fixed open proper subset $U \subset M$ to increasing neighborhoods $Z_{1/\epsilon} \subset Z$ such that $g_\epsilon|_U \cong \left(\epsilon^2 (\phi_\epsilon)^* g|_{Z_{1/\epsilon}}\right)|_U$. Moreover, on $M - U$, $g_\epsilon \to g_0$ smoothly as $\epsilon \to 0$ and any $K \subset\subset M_0^0$ is diffeomorphic to some fixed $K' \subset M$ so that $g_\epsilon \to g_0$ smoothly and uniformly on K'.*

Proof

The existence of ϕ_ϵ and $U \subset M$ follows immediately from the definition of resolution blowup and the diffeomorphism between the resolution blowup M and $\{xr = \epsilon\} \subset \mathcal{S}$. By the above remarks, on the neighborhood $U \subset M$ where this diffeomorphism is defined, $g_\epsilon|_U = (\epsilon^2)(\phi_\epsilon)^* g_z|_{Z_{1/\epsilon}}$.

Since $g_\epsilon = g_0 + \mathcal{O}(\epsilon^2)$, the smooth convergence of g_ϵ to g_0 on $M - U$ follows immediately. Any compact subset $K \subset\subset M_0$ is contained in $M_{0,\epsilon}$ for some $\epsilon > 0$ and so is diffeomorphic to $K_\epsilon \subset\subset M_\epsilon$ and to $K' \subset\subset (M - U)$. Conversely, any $K \subset\subset (M - U)$ is diffeomorphic to $K_\epsilon \subset M_\epsilon$ and to $K' \subset\subset M_{0,\epsilon} \subset M_0$.

♡

3 Geometric Preliminaries

This section is a brief review of the theory and terminology of manifolds with corners, b maps, and blowups. A complete reference is [26], see also [23].

3.1 Manifolds with Corners

Let X be a manifold with corners. This means that near any of its points, X is modeled on a product $[0, \infty)^k \times \mathbb{R}^{n-k}$ where k depends on the point and is the maximal codimension of the boundary face containing that point. We also assume that all boundary faces of X are embedded so they too are manifolds with corners. The space $\mathcal{V}(X)$ of all smooth vector fields on X is a Lie algebra under the standard bracket operation. It contains the Lie subalgebra

$$\mathcal{V}_b(X) := \{V \in \mathcal{V}(X)\,;\, V \text{ is tangent to each boundary face of } X.\} \tag{1}$$

Then $\mathcal{V}_b(X)$ is itself the space of all smooth sections of a vector bundle,

$$\mathcal{V}_b(X) = \mathcal{C}^\infty(X; {}^bTX),$$

where ${}^bTX$ is the bundle defined so that the above holds and is called the b-tangent bundle.

3.1.1 Blowing Up

An embedded codimension k submanifold Y of a manifold with corners X is called a p-submanifold (p for product) if near each point of Y there are local product coordinates so that Y is defined by the vanishing of some subset of them. In other words, X and Y must have consistent local product decompositions. Then one can define a new manifold with corners $[X; Y]$ to be the normal blowup of X around Y. This is obtained by replacing Y by its inward-pointing spherical normal bundle. The union of this normal bundle and $X - Y$ has a unique minimal differential structure as a manifold with corners so that the lifts of smooth functions on X and polar coordinates around Y are smooth. One can also consider iterated blowups, written $[[X; Y_1]; Y_2]]$, where Y_1 is a p-submanifold of X and Y_2 is a p-submanifold of Y_1. We may consider any finite sequence of such blowups. If we have such a sequence of embedded p-submanifolds,

$$X \supset Y_1 \supset Y_2 \supset Y_3 \ldots \supset Y_n$$

then the iterated blowup

$$[[X; Y_1]; Y_2]; \dots; Y_n]$$

can be performed in any order with the same result [24]. Blowups may also be defined using equivalence classes of curves [26]. Let r be a defining function for the p submanifold and consider the family of curves $\gamma(t) = (r(t), y(t))$ such that

$$\gamma(t) \in Y \iff t = 0,$$

$$r(t) = O(t).$$

Let E be the set of equivalence classes of all such curves with

$$\gamma \sim \gamma' \iff (y - y')(t) = O(t),\ (r - r')(t) = O(t^2).$$

There is a natural $\mathbb{R}^+$ action on E given by

$$\mathbb{R}^+ \ni a : \gamma(t) \to \gamma(at).$$

Then E modulo this equivalence relation is naturally diffeomorphic to $N^+(Y)$, the inward pointing spherical normal bundle of Y, so we can define $[X; Y]$ by

$$[X; Y] = (X - Y) \cup E/(\mathbb{R}^+ - \{0\}).$$

We can also define *parabolic* blowups in certain contexts [9]. Let Y be a p-submanifold of codimension k so that there exist local coordinates $(r, y) = (r_1, \dots, r_k, y_1, \dots, y_{n-k})$ in a neighborhood of Y with r_i vanishing precisely at Y, and so that dr_1 induces a sub-budle of the tangent bundle TX. Instead of the above equivalence classes of curves we consider $\gamma(t)$ such that

$$\gamma(t) = (r_1(t), \dots, r_k(t), y_1(t), \dots y_{n-k}(t)) \in Y \iff t = 0,$$

$$r_i(t) = O(t),\ i \neq 1,\ r_1(t) = O(t^2).$$

Two such curves are equivalent if

$$\gamma \sim \gamma' \iff (y_j - y_j')(t) = O(t), (r_i - r_i')(t) = O(t^2)\, i \neq 1,\ (r_1(t) - r_1'(t)) = O(t^3).$$

Since dr_1 is a sub-bundle of TX there is a natural $\mathbb{R}^+$ action on the set of equivalence classes E_2 of all such curves,

$$\mathbb{R}^+ \ni a : \gamma(t) \to (r_1(a^2 t), r_i(at), \dots, y_j(at)).$$

The set of equivalence classes of all such curves modulo this $\mathbb{R}^+$ action is naturally diffeomorphic to the inward pointing r_1-parabolic normal bundle of Y,

$$E_2/(\mathbb{R}^+ - \{0\}) \cong PN^+_{r_1}(Y).$$

We define the r_1-parabolic blowup of X around Y as the union of $X - Y$ and this inward pointing r_1-parabolic bundle,

$$[X; Y, dr_1] := (X - Y) \cup PN^+_{r_1}(Y).$$

The union of this r_1-parabolic bundle and $X - Y$ again has a unique minimal differential structure as a manifold with corners so that the lifts of smooth functions on X and r_1-parabolic coordinates around Y are smooth. By r_1-parabolic coordinates around Y, we mean the coordinates,

$$\rho = (r_1^2 + r_2^4 + \ldots + r_k^4)^{\frac{1}{4}},\ \theta = (\theta_1, \ldots \theta_n) \in \mathbb{S}^{n-1},$$

with local coordinates $(r_1, \ldots r_k, y_1, \ldots, y_{n-k})$ in a neighborhood of Y satisfying

$$r_i = \rho\theta_i,\ i \neq 1,\ r_1 = \rho^2\theta_1, \quad y_j = \rho\theta_j.$$

For any parabolic or spherical blowup there is a natural blow-down map $\beta_* : [X; Y] \to X$ and corresponding blow-up map $\beta^* : X \to [X; Y]$, so that the image of Y under β^* is a boundary hypersurface of $[X; Y]$ diffeomorphic to the inward pointing spherical (or parabolic) normal bundle of Y. As such,

$$[X; Y] = (X - Y) \cup \beta^*(Y).$$

3.1.2 b-Maps and b-Fibrations

Definition 7. *Let M_1 be a manifold with boundary hypersurfaces, $\{N_j\}_{j=1}^k$, and defining functions r_j. Let M_2 be a manifold with boundary hypersurfaces, $\{L_i\}_{i=1}^l$, and defining functions ρ_i. Then $f : M_1 \to M_2$ is called a* b-map *if for every i there exist nonnegative integers $e(i, j)$ and a smooth nonvanishing function h such that $f^*(\rho_i) = h \prod_{j=1}^k r_j^{e(i,j)}$.*

The image under a b-map of the interior of each boundary hypersurface of M_1 is either contained in or disjoint from each boundary hypersurface of M_2 and the order of vanishing of the differential of f is constant along each boundary hypersurface of M_1. The matrix $(e(i, j))$ is called the lifting matrix for f.

In order for the map, f, to preserve polyhomogeneity, stronger conditions are required. Associated to a manifold with corners are the b-tangent and cotangent bundles, ${}^bTM$ (1) and ${}^bT^*M$.[6] The map f may be extended to induce the map ${}^bf_* :{}^b TM_1 \to{}^b TM_2$.

Definition 8. *The b-map, $f : M_1 \to M_2$, is called a* b-fibration *if the associated maps ${}^bf_*$ at each $p \in \partial M_1$ are surjective at each $p \in \partial M_1$ and the lifting matrix $(e(i, j))$ has the property that for each j there is at most one i such that $(e(i, j)) \neq 0$. In other words, f does not map any boundary hypersurface of M_1 to a corner of M_2.*

[6] These are also called the totally characteristic tangent and cotangent bundles.

3.1.3 b-manifolds and the b-blowup

A b-manifold is a manifold with corners that is closely related to conic manifolds and ac scattering metrics.

Definition 9. *Let (X, g) be a smooth Riemannian manifold with boundary (Y, h) and boundary defining function x such that in a collared neighborhood N of the boundary X has a product decomposition, $N \cong [0, x_1)_x \times Y$ and in this neighborhood*

$$g = \frac{dx^2}{x^2} + h(x),$$

*where $h(x)$ is a smoothly varying family of metrics on Y that converges smoothly to h as $x \to 0$. Then (X, g) is said to be a b-*manifold.

Equivalently, a b-manifold is a complete manifold with asymptotically cylindrical ends. The Schwartz kernels of operators on a b-manifold with reasonable regularity lift to a blown up manifold called the b-double space. This space is obtained from X^2 by performing a radial blowup called the b-blowup along the codimension 2 corner at the boundary in each copy of X and it is written X_b^2,

$$X_b^2 = [X \times X; \partial X \times \partial X] = [X \times X; Y \times Y]. \tag{2}$$

For any manifold M with boundary having a product structure in a neighborhood of the boundary, we may define the b-blowup in the analogous way, $M_b^2 := [M \times M; \partial M \times \partial M]$.

3.2 Asymptotically conic convergence double space

The acc double space is an instructive model for the more complicated acc heat space in section 7. Let

$$\mathcal{S}_b^2 := [\mathcal{S}^2; Y \times Y],$$

The acc double space $\mathcal{D}$ is the submanifold of $\mathcal{S}_b^2$ defined by the vanishing set of $f(p) = x(p)r(p) - x'(p)r'(p) = \epsilon(p) - \epsilon'(p)$:

$$\mathcal{D} = \{p \in \mathcal{S}_b^2 : f(p) = x(p)r(p) - x'(p)r'(p) = 0\} = \{p \in \mathcal{S}_b^2 : \epsilon(p) = \epsilon'(p)\}.$$

The acc double space has various boundary faces but we are only interested in those at $\epsilon = 0$. There are four boundary faces at $\epsilon = 0$, described in the following table. Here and throughout, we label each face F_{wxyz} where the subscript indicates the order to which each of the scalar variables x, r, x', r' vanishes at that face.

Arising from	face	geometry
$x = 0, x' = 0,$	F_{1010}	$[\bar{Z} \times \bar{Z}; Y \times Y]$
$x = 0, r' = 0$	F_{1001}	$[\bar{Z} \times M_0; Y \times Y]$
$r = 0, x' = 0$	F_{0110}	$[M_0 \times \bar{Z}; Y \times Y]$
$r = 0, r' = 0$	F_{0101}	$[M_0 \times M_0; Y \times Y]$

To see that $\mathcal{D}$ is a smooth submanifold of $\mathcal{S}_b^2$ we consider the function

$$f(p) = x(p)r(p) - x'(p)r'(p). \tag{3}$$

Away from the boundary faces f is smooth with non-vanishing differential. In a neighborhood of $S_{11} - F_{1001}$, let

$$f_{0110}(p) = \frac{x(p)}{x'(p)} - \frac{r'(p)}{r(p)}.$$

Since $x' r f_{0110} = f$, we see that f_{0110} is smooth near the $\epsilon = 0$ boundary faces away from where those faces meet F_{0110}. Moreover, wherever defined, f_{0110} has nonvanishing differential and the zero set of f_{0110} coincides with that of f away from F_{0110}. Similarly, let

$$f_{1001}(p) = \frac{x(p)}{r'(p)} - \frac{x'(p)}{r(p)}.$$

f_{1001} is smooth with nonvanishing differential and has the same vanishing set as f in a neighborhood of $\{\epsilon = 0\} - F_{1001}$. This shows that $\mathcal{D}$ is a smooth submanifold of $\mathcal{S}_b^2$. While the acc double space will not be used here, we note that the acc double space, with an additional blowup along the diagonal for $\epsilon \geq 0$, would be the natural space on which to study the resolvent behavior under ac convergence.

4 Analytic Preliminaries

Since we are working on manifolds with singularities, corners and boundaries, we briefly review some key features of the analysis in these settings.

4.1 Polyhomogeneous conormal functions

On manifolds M with corners having a consistent local product structure near each boundary and corner, a natural class of functions (or sections) with good regularity near the boundary and corners are the polyhomogeneous conormal functions (or sections). For a complete reference on polyhomogeneity on manifolds with corners, see [23]. In a neighborhood of a corner, we have coordinates $(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ where $x_1, \ldots, x_k$ vanish at this corner and $(y_1, \ldots, y_{n-k})$ are smooth local coordinates on a smooth compact $n - k$ manifold Y. The edge tangent bundle $\mathcal{V}_e$ in a neighborhood of this corner is spanned over $\mathcal{C}^\infty(M)$ by the vector fields,

$$\{x_i \partial_{x_i}, \partial_{y^\alpha}\}.$$

The basic conormal space of sections is

$$\mathcal{A}^0(M_0) = \{\phi : V_1 ... V_l \phi \in L^\infty(M_0), \forall\, V_i \in \mathcal{V}_e, \text{ and } \forall\, l\}.$$

Let α and p be multi indices with $\alpha_j \in \mathbb{C}$ and $p_j \in \mathbb{N}_0$. Then we define

$$\mathcal{A}^{\alpha,p}(M_0) = x^\alpha (\log x)^p \mathcal{A}^0.$$

The space $\mathcal{A}^*$ is the union of all these spaces, for all α and p. The space $\mathcal{A}^*_{phg}(M_0)$ consists of all conormal distributional sections which have an expansion of the form

$$\phi \sim \sum_{Re(\alpha_j)\to\infty} \sum_{p=0}^{p_j} x^{\alpha_j} (\log x)^p a_{j,p}(x,y),\ a_{j,p} \in \mathcal{C}^\infty.$$

We define an index set to be a discrete subset $E \subset \mathbb{C} \times \mathbb{N}_0$ such that

$$(\alpha_j, p_j) \in E, \quad |(\alpha_j, p_j)| \to \infty \implies Re(\alpha_j) \to \infty.$$

Then, the space $\mathcal{A}^E_{phg}(M_0)$ consists of those distributional sections $\phi \in \mathcal{A}^*_{phg}$ having polyhomogeneous expansions with $(\alpha_j, p_j) \in E$.

4.2 Conic differential operators and b-operators

Let (M_0, g_0) be a Riemannian manifold with isolated conic singularity, defined by $x = 0$, so that in a neighborhood of the singularity,

$$M_0 \cong (0, x_1)_x \times Y, \quad g_0 = dx^2 + x^2 h(x)$$

with $h(x) \to h$, where (Y, h) is a smooth, $n-1$ dimensional compact manifold. A conic differential operator of order m is a smooth differential operator on M_0 such that in a neighborhood of the singularity it can be expressed

$$A = x^{-m} \sum_{k=0}^{m} B_k(x)(-x\partial_x)^k$$

with $B_k \in \mathcal{C}^\infty((0, x_1), \mathrm{Diff}^{m-k}(Y))$, where $\mathrm{Diff}^j(Y)$ denotes the space of differential operators of order $j \in \mathbb{N}_0$ on Y with smooth coefficients. The cone differential operators are elements of the cone operator calculus; for a detailed description, see [19]. These cone operators are closely related to b-operators. A b-operator of order m is a smooth differential operator such that near the boundary it can be expressed

$$A = \sum_{k=0}^{m} B_k(x)(-x\partial_x)^k$$

with $B_k \in \mathcal{C}^\infty((0, x_1), \mathrm{Diff}^{m-k}(Y))$. We see that a cone differential operator of order m is equal to a rescaled b-differential operator of order m. In other words, if A is an order

m cone differential operator then $x^m A$ is a b-differential operator. In local coordinates $(x, y_1, \ldots, y_{n-1})$ near the boundary of M_0, a b-operator may be expressed as

$$A = \sum_{j+|\alpha|\leq m} a_{j,\alpha}(x,y)(-x\partial_x)^j(\partial_{y^\alpha}).$$

The b-symbol of A is

$${}^b\sigma_m(A) = \sum_{j+|\alpha|=m} a_{j,\alpha}(x,y)\lambda^j\eta^\alpha.$$

Here λ and η are linear functions on ${}^bT^*M_0$ defined by the coordinates so that a generic element of ${}^bT^*M_0$ is

$$\lambda\frac{dx}{x} + \sum_{i=1}^{n}\eta_i dy_i.$$

The b-operator is b-elliptic if the symbol ${}^b\sigma_m(A) \neq 0$ on ${}^bT^*M_0 - \{0\}$.

The scalar Laplacian on M_0 is

$$x^{-2}\{(-x\partial_x)^2 + (-n+1+xH^{-1}(\partial_x H)(-x\partial_x)) + \Delta_{h(x)}\} = x^{-2}L_b$$

where L_b is an elliptic order 2 b-operator and H is a smooth function depending on the metric. Similarly, a geometric Laplacian Δ_0 on M_0 is also of the form

$$\Delta_0 = x^{-2}L_b,$$

for an elliptic order two b-operator acting on sections of the vector bundle. The Schwartz kernel of L_b is a distribution on the b-double space $M^2_{0,b}$. By the b-calculus theory, (see [26]) L_b has a parametrix G_b, such that G_b is a b-operator of order -2 with

$$G_b L_b = I - R$$

where I is the identity operator and R is a b-operator with polyhomogeneous Schwartz kernel on the b-double space. Then, for any $u \in \mathcal{L}^2(x^{n-1}dxdy)$ with $\Delta_0 u = f \in \mathcal{L}^2(x^{n-1}dxdy)$

$$(x^2G_b)(x^{-2}L_b u) = (x^2G_b)f = u - Ru \qquad \Longrightarrow\ u = x^2G_b f + Ru = \alpha + \beta.$$

The first term, $\alpha \in x^2H_b^2 \subset x^2\mathcal{L}^2(x^{n-1}dxdy)$. The second term $\beta \in \mathcal{L}^2(x^{n-1}dxdy)$ has a polyhomogeneous expansion as $x \to 0$,

$$\beta \sim \sum_{j=0}^{\infty}\sum_{k=0}^{N_j} x^{\gamma_j+k}\varphi_j(y).$$

Above γ_j is an indicial root for the operator L_b and φ_j is an eigensection for the induced geometric Laplacian on (Y, h). Then,

$$u = \alpha + \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} x^{\gamma_j + k} \varphi_j(y) \tag{4}$$

where $\alpha \in x^2 \mathcal{L}^2(x^{n-1} dx dy)$. This decomposition plays a key role in the proof of spectral convergence.

4.3 Friedrich's domain of the conic Laplacian

A geometric Laplacian Δ_0 on a conic manifold is an unbounded operator on $\mathcal{L}^2$ sections of the bundle. It can be extended to various domains in $\mathcal{L}^2$; the minimal domain $\mathcal{D}_{\min}$ is the $\mathcal{L}^2$ closure of the graph of Δ_0 over $\mathcal{C}_0^\infty$. The largest domain $\mathcal{D}_{\max}$ is the $\mathcal{L}^2$ closure of the graph of Δ_0 over $\mathcal{L}^2$. Each of these domains are dense in $\mathcal{L}^2(M_0)$, and the extension of the Laplacian to either domain is a closed operator. On complete manifolds $\mathcal{D}_{\min} = \mathcal{D}_{\max}$ by the Gaffney-Stokes Theorem [11]. However, M_0 is incomplete and so for a general geometric Laplacian these domains will not be equal. The Friedrich's domain $\mathcal{D}_F$ lies between $\mathcal{D}_{\min}$ and $\mathcal{D}_{\max}$ and is the closure of the graph of Δ_0 in $\mathcal{L}^2$ with respect to the densely defined Hermitian form,

$$Q(u, v) = \int_{M_0} \langle \nabla u, \nabla v \rangle.$$

The extension of the Laplacian to the Friedrich's domain, known as the Friedrich's extension of the Laplacian, preserves the lower bound and is essentially self adjoint. Here, we work exclusively with the Friedrich's extension of the Laplacian.

For elements of $\mathcal{D}_{max}$, with $u \in \mathcal{L}^2$ and $\Delta_0 u = f \in \mathcal{L}^2$, we have the expansion (4) from the preceding section,

$$u = \alpha + \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} x^{\gamma_j + k} \varphi_j(y).$$

The volume form on M_0 near the singularity is asymptotic to $x^{n-1} dx dy$. Therefore, the exponents γ_j must all be strictly greater than $-\frac{n}{2}$. For $v \in \mathcal{D}_{\min} \subset \mathcal{D}_{max}$ the decomposition (4) and the definition of $\mathcal{D}_{min}$ imply that $\mathcal{D}_{\min} \subset x^2 \mathcal{L}^2$. The equality of $\mathcal{D}_{\min}$ and $\mathcal{D}_{\max}$ then depends on the indicial roots of $L_b = x^2 \Delta_0$. For further discussion of domains of the conic Laplacian, see [13], whose results include:

$$\mathcal{D}_F = \{f \in \mathcal{L}^2 : \Delta_0 f \in \mathcal{L}^2 \text{ and } f = \mathcal{O}(x^{\frac{2-n+\delta}{2}}) \text{ as } x \to 0, \text{ for some } \delta > 0\}.$$

We will use this characterization of the domain of the (Friedrich's extension of the) Laplacian in the proof of the first theorem.

5 Spectral Convergence

We now have all the necessary ingredients to prove spectral convergence.

Theorem 1. *Let (M_0, g_0) be a compact Riemannian n-manifold with isolated conic singularity, and let (Z, g_z) be an asymptotically conic space, with $n \geq 3$.*[7] *Assume (M, g_ϵ) converges asymptotically conically to (M_0, g_0). Let (E_0, ∇_0) and (E_z, ∇_z) be Hermitian vector bundles over (M_0, g_0) and (Z, g_z), respectively, so that each of these bundles in a neighborhood of the boundary is the pullback from a bundle over the cross section (Y, h). Let Δ_0, Δ_z be the corresponding Friedrich's extensions of geometric Laplacians, and let Δ_ϵ be the induced geometric Laplacian on (M, g_ϵ). Assume Δ_z has no $\mathcal{L}^2$ nullspace. Then the accumulation points of the spectrum of Δ_ϵ as $\epsilon \to 0$ are precisely the points of the spectrum of Δ_0, counting multiplicity.*

The theorem follows from the following three statements: the inclusion *accumulation* $\sigma(\Delta_\epsilon) \subset \sigma(\Delta_0)$, the reverse inclusion *accumulation* $\sigma(\Delta_\epsilon) \supset \sigma(\Delta_0)$, and correct multiplicities.

5.1 Accumulation $\sigma(\Delta_\epsilon) \subset \sigma(\Delta_0)$

We extract a smoothly convergent sequence of eigensections corresponding to a converging sequence of eigenvalues as $\epsilon \to 0$, and show that the limit section of this sequence is an eigensection for the conic metric and its eigenvalue is the accumulation point. For this argument, we work with sequences of metrics $\{g_{\epsilon_j}\}$ which we abbreviate $\{g_j\}$ with Laplacians Δ_j.

Let $\lambda(\epsilon_j)$ be an eigenvalue of Δ_j, with eigensection f_j. Assume that $\lambda(\epsilon_j) \to \bar{\lambda}$. Over any compact set $K \subset M_0^0$, the metric $g_j = g_{\epsilon_j}$ converges smoothly to g_0 by Lemma 1, thus so do the coefficients of $\Delta_j - \Delta(\epsilon_j)$. Hence, normalizing f_j by $\sup_M |f_j| = 1$, it follows using standard elliptic estimates and the Arzela-Ascoli theorem that f_j converges in $\mathcal{C}^\infty$ on any compact subset of M_0^0. Furthermore the limit section $\bar{f}$ satisfies the limiting equation

$$\Delta_0 \bar{f} = \bar{\lambda} \bar{f}.$$

However, we do not know yet that $\bar{f} \not\equiv 0$, nor, even if this limit is nontrivial, that it lies in the domain of the Friedrichs extension of Δ_0. This is the content of the arguments below.

5.1.1 Weight Functions

Let $\phi_\epsilon : M_{0,\epsilon} - M_{0,\delta} \to Z_{1/\epsilon} - Z_{1/\delta}$ as in definition 4. We identify $Z_{1/\delta}$ with a fixed $K \subset U \subset M$ so that $M_{0,\epsilon} - M_{0,\delta} \cong (U - K)$, $K \cong Z_{1/\delta}$.

[7] In this theorem, unlike the main theorem with heat kernel convergence, we require the dimension be at least 3 due to the $\mathcal{L}^2$ regularity of the Friedrich's domain of the Laplacian on the conic manifold.

Let

$$w_\epsilon = \begin{cases} c & \text{on } M - U. \\ \epsilon\,(\phi_\epsilon^{-1})^*\rho & \text{on } U - K. \\ c\epsilon & \text{on } K. \end{cases}$$

Above, c is a constant and no generality is lost by assuming $c = 1$. Let $w_j = w_{\epsilon_j}$. For some $\delta > 0$ to be chosen later, replacing f_j by $\frac{f_j}{||w_j^\delta f_j||_\infty}$ we assume the supremum of $|f_j w_j^\delta|$ is 1 on M. Since M is compact, $|f_j|$ attains a maximum at some point $p_j \in M$, and we may assume p_j converges to some $\bar{p} \in M$. The argument splits into three cases depending on how and where p_j accumulates in M.

5.1.2 Case 1: $w_j(p_j) \to c > 0$ as $j \to \infty$.

In this case, the points $\{p_j\}$ accumulate in a compact subset of $M - U$ which we may identify with a compact subset of M_0^0 by the lemma. So, we may assume that these points converge to some point $\bar{p} \neq p$ (the singularity). The maximum of $|f_j w_j^\delta|$ on M is 1 and occurs at p_j so

$$|f_j| \leq w_j^{-\delta} \text{ on } M \text{ for each } j \implies |f_j(p_j)| \to c^{-\delta} \text{ as } j \to \infty.$$

The locally uniform $\mathcal{C}^\infty$ convergence of f_j to $\bar{f}$ implies that $|\bar{f}|$ satisfies a similar bound,

$$|\bar{f}| \leq x^{-\delta} \text{ as } x \to 0,$$

and clearly $|\bar{f}(\bar{p})| = c^{-\delta} \neq 0$. By the dimension assumption $n \geq 3$ and the characterization of the Friedrich's Domain of the Laplacian, we may choose δ so that

$$\frac{2-n}{2} < -\delta < 0.$$

Then $\bar{f}$ lies in the Friedrich's domain of the Laplacian and satisfies

$$\Delta_0 \bar{f} = \bar{\lambda}\bar{f},$$

so $\bar{\lambda}$ is an eigenvalue of Δ_0.

5.1.3 Case 2: $|w_j(p_j)| \leq c(\epsilon_j)$ as $j \to \infty$.

Analysis on Z in this case leads to a contradiction. Let $\phi_j = \phi_{\epsilon_j}$, $\tilde{f}_j = f_j(\phi_j^{-1})$. Let $\tilde{p_j} = \phi_j(p_j)$. Because $|f_j w_j^\delta|$ attains its maximum value of 1 at p_j, $|\tilde{f}_j(\tilde{p_j})| = (w_j(p_j))^{-\delta}$. Rescale f_j and $\tilde{f}_j$, replacing them respectively with $(w_j(p_j))^\delta f_j$ and $(w_j(p_j))^\delta \tilde{f}_j$ so that the maximum of $|\tilde{f}_j \rho^\delta|$ occurs at the point $\tilde{p_j} \in Z_j$ and is equal to 1. Since $w_j(p_j) = \mathcal{O}(\epsilon_j)$, $\rho(\tilde{p_j}) = \epsilon_j^{-1} w_j(p_j)$ stays bounded for all j, and so we assume $\tilde{p_j}$ converges to $\tilde{p} \in Z$.

By Lemma 1, $(Z_j, \epsilon_j^{-2}\phi_j^* g_j|_U)$ converges smoothly to (Z_j, g_Z). This implies the following equation is satisfied by $\tilde{f}_j$ on Z_j,

$$\Delta_Z \tilde{f}_j = \epsilon_j^2 \lambda(\epsilon_j)\tilde{f}_j + O(\epsilon_j).$$

Since the $\lambda(\epsilon_j)$ are converging to $\bar{\lambda}$ and $|\tilde{f}_j \rho^\delta| \leq 1$ on Z_j, we have

$$\Delta_Z \tilde{f}_j \to 0 \text{ as } j \to \infty, \text{ on any compact subset of } Z.$$

This implies $f_j \to \bar{f}$ on M and correspondingly, $\tilde{f}_j \to \tilde{f}$ locally uniformly $\mathcal{C}^\infty$ on Z and $\tilde{f}$ satisfies

$$\Delta_Z \tilde{f} = 0, \quad |\tilde{f}\rho^\delta| \leq 1,$$

where equality holds in the second equation at the point p. This shows that $\tilde{f}$ is not identically zero on Z and $\tilde{f} = O(\rho^{-\delta})$ as $\rho \to \infty$. Since $\tilde{f}$ is smooth on any compact subset of Z and is therefore in $\mathcal{L}^2_{loc}(Z)$, choosing $\delta > n-2$ contradicts the assumption that Z has no $\mathcal{L}^2$ nullspace.

5.1.4 Case 3: $w_j(p_j) \to 0, \quad \frac{\epsilon_j}{w_j(p_j)} \to 0$ as $j \to \infty$.

In this case, the points $\phi_j(p_j) \to \infty$ in Z, so we rescale and derive a contradiction on the complete cone over (Y, h). Consider the coordinates (ρ, y) on Z defined for $\rho \geq \rho_1$. In these coordinates $g_z = d\rho^2 + \rho^2 h(\rho)$. Let $r_j = \frac{\epsilon_j}{w_j(p_j)}\rho$ and $\tilde{g}_j$ on Z_j be defined by

$$\tilde{g}_j = \left(\frac{\epsilon_j}{w_j(p_j)}\right)^2 g_Z.$$

Then,

$$(Z_j, \tilde{g}_j) \cong \left(\left(\frac{\rho_1 \epsilon_j}{w_j(p_j)}, \frac{1}{w_j(p_j)}\right) \times Y,\ dr_j^2 + r_j^2 h(r_j\epsilon_j/w_j(p_j)\right).$$

As $j \to \infty$, $h(r_j\epsilon_j/w_j(p_j))$ converges smoothly to h, and

$$\tilde{g}_j \to g_C = dr^2 + r^2 h$$

on the complete cone C over (Y, h). Let $\tilde{f}_j = w_j(p_j)^\delta f_j(\phi_j^{-1})$. Since $|f_j w_j^\delta| \leq 1$ with equality at p_j,

$$|\tilde{f}_j r_j^\delta| \leq 1 \text{ on } (Z_j, \tilde{g}_j) \text{ with equality at } \tilde{p}_j = \phi_j(p_j).$$

Let $\tilde{\Delta}_j$ on Z_j be the Laplacian induced by $\tilde{g}_j$ on Z_j,

$$\tilde{\Delta}_j = \frac{w_j(p_j)^2}{\epsilon^2}\Delta_Z,$$

so

$$\tilde{\Delta}_j \tilde{f}_j = \epsilon_j^2 \frac{w_j(p_j)^2}{\epsilon_j^2} \lambda(\epsilon_j) \tilde{f}_j + O(\epsilon_j) \text{ on } (Z_j, \tilde{g}_j).$$

Since $w_j(p_j) \to 0$ as $j \to \infty$, there is a locally uniform $\mathcal{C}^\infty$ limit f_c of $\{\tilde{f}_j\}$ on C which satisfies

$$|f_c r^\delta| \leq 1, \quad \Delta_c f_c = 0.$$

Since the points $\tilde{p_j}$ stay at a bounded radial distance with respect to the radial variable r_j on Z_j, we may assume $\tilde{p_j} \to p_c$ for some $p_c \in C$. At this point, $|f_c(p_c) r(p_c)^\delta| = 1$ so f_c is not identically zero. By separation of variables (see, for example, [22]), f_c has an expansion in an orthonormal eigenbasis $\{\phi_j\}$ of $\mathcal{L}^2(Y, h)$,

$$f_c = \sum_{j \geq 0} a_{j,+} r^{\gamma_{j,+}} \phi_j(y) + a_{j,-} r^{\gamma_{j,-}} \phi_j(y)$$

where $\gamma_{j,+/-}$ are indicial roots corresponding to ϕ_j and $a_{j,+/-} \in \mathbb{C}$. In order for $|f_c r^\delta| \leq 1$ globally on C, we must have only one term in this expansion, $f_c = a_j r^{-\delta} \phi_j(y)$. Because the indicial roots are discrete, we may choose δ so that $-\delta$ is not an indicial root. This is a contradiction.

5.2 $\sigma(\Delta_0) \subset$ Accumulation $\sigma(\Delta_\epsilon)$

We use the Rayleigh-Ritz characterization of the eigenvalues [2]. Let $\lambda_l(\epsilon_j)$ be the l^{th} eigenvalue of Δ_j and let

$$R_j(f) := \frac{\langle \nabla f, \nabla f \rangle_j}{\langle f, f \rangle_j}.$$

The subscript j indicates that the inner product is taken with respect to the $\mathcal{L}^2$ norm on M with the g_j metric. The eigenvalues are characterized using Mini-Max by

$$\lambda_l(\epsilon_j) = \inf_{\dim\, L = l,\, L \subset \mathcal{C}^1(M)} \sup_{f \in L, f \neq 0} R_j(f).$$

Similarly this characterization holds for the eigenvalues of the (Friedrich's extension of the) conic Laplacian which are known to be discrete (see [4], for example). Because $\mathcal{C}_0^\infty(M_0)$ is dense in $\mathcal{L}^2(M_0)$ we may restrict to subspaces contained in $\mathcal{C}_0^\infty(M_0)$. Then, the l^{th} eigenvalue of Δ_0 is

$$\bar{\lambda}_l = \inf_{\dim\, L = l,\, L \subset \mathcal{C}_0^\infty(M_0)} \sup_{f \in L, f \neq 0} R_0(f).$$

Let $\bar{\lambda}_l$ be the l^{th} eigenvalue in the spectrum of Δ_0. Fix $\epsilon > 0$. Then there exists $L \subset \mathcal{C}_0^\infty$ with $dim(L) = l$ and

$$\sup_{f \in L, f \neq 0} R_0(f) < \bar{\lambda}_l + \epsilon.$$

Since any $f \in L$ is also in $\mathcal{C}_0^\infty(M)$ and because L is finite dimensional, by the local convergence of g_j to g_0, for large j

$$|R_j(f) - R_0(f)| < \epsilon \text{ for any } f \in L.$$

Since $\lambda_l(\epsilon_j)$ is the infimum

$$\lambda_l(\epsilon_j) \leq \bar{\lambda}_l + 2\epsilon.$$

This shows $\{\lambda_l(\epsilon_j)\}$ is bounded in j, and so we extract a convergent subsequence and a corresponding convergent sequence of eigensections which exists by the preceding arguments. For each l we take

$$\lambda_l(\epsilon_j) \to \mu_l \leq \bar{\lambda}_l,$$

$$f_{j,l} \to u_l, \quad \Delta_0 u_l = \mu_l u_l.$$

These limit eigensections u_l are seen to be orthogonal as follows. Fix l, k, with $f_{j,k} \to u_k$ and $f_{j,l} \to u_l$. Since $\mathcal{C}_0^\infty(M_0)$ is dense in $\mathcal{L}^2(M_0)$ we may choose a smooth cutoff function χ vanishing identically near the singularity in M_0 such that

$$||\chi u_k - u_k||_{L^2(M_0)} < \epsilon,$$

$$||\chi u_l - u_l||_{L^2(M_0)} < \epsilon,$$

$$Vol_j(M \, - \, supp(\chi)) < \epsilon.$$

Then on the support of χ, $g_j \to g_0$ uniformly so for large j,

$$|\langle u_k, u_l\rangle_0 - \langle \chi u_k, \chi u_l\rangle_0| < \epsilon,$$

$$|\langle \chi u_k, \chi u_l\rangle_0 - \langle \chi u_k, \chi u_l\rangle_j| < \epsilon,$$

$$|\langle \chi u_k, \chi u_l\rangle_j - \langle \chi u_k, \chi f_{j,l}\rangle_j| < \epsilon,$$

$$|\langle \chi u_k, \chi f_{j,l}\rangle_j - \langle \chi f_{j,k}, \chi f_{j,l}\rangle_j| < \epsilon.$$

Since the eigensections for Δ_j were chosen to be orthonormal and the volume of $(M - \text{support}(\chi))$ is small with respect to g_j,

$$|\langle \chi f_{j,k}, \chi f_{j,l}\rangle_j| < 2\epsilon.$$

Thus, $\langle u_k, u_l\rangle_0$ can be made arbitrarily small and u_k, u_l are orthogonal for $l \neq k$. We complete this basis to form an eigenbasis of $\mathcal{L}^2(M_0)$. Let $\bar{f}_l$ be an arbitrary element of this eigenbasis, with eigenvalue $\bar{\lambda}_l$. We wish to show that this $\bar{f}_l$ is actually the u_l above, defined to be the limit of (a subsequence of) $\{f_{j,l}\}$, and hence the corresponding μ_l is equal to $\bar{\lambda}_l$. Again, assume the smooth cut-off function χ is chosen so that

$$||\chi \bar{f}_l - \bar{f}_l||_{L^2(M_0)} < \epsilon.$$

For each j we expand $\chi \bar{f}_l$ in eigensections of Δ_j,

$$\chi \bar{f}_l = \sum_{k=0}^{\infty} a_{j,k} f_{j,k}, \quad \text{where } a_{j,k} = \langle \chi \bar{f}_l, f_{j,k} \rangle_j.$$

Now, fix k and choose χ such that

$$||\chi u_k - u_k||_{L^2(M_0)} < \epsilon.$$

Then,

$$\begin{aligned} |\langle \chi \bar{f}_l, f_{j,k} \rangle_0 - \langle \chi \bar{f}_l, f_{j,k} \rangle_j| &< \epsilon, \\ |\langle \chi \bar{f}_l, f_{j,k} \rangle_j - \langle \chi \bar{f}_l, u_k \rangle_j| &< \epsilon, \\ |\langle \langle \chi \bar{f}_l, u_k \rangle_j - \langle \chi \bar{f}_l, u_k \rangle_0| &< \epsilon, \\ |\langle \chi \bar{f}_l, u_k \rangle_0 - \langle \bar{f}_l, u_k \rangle_0| &< \epsilon. \end{aligned}$$

By the orthogonality $\langle \bar{f}_l, u_k \rangle_0 = 0$ if $\bar{f}_l \neq u_k$, and otherwise is 1, so for each k, $a_{j,k} \to 0$ as $j \to \infty$, for all k with $u_k \neq \bar{f}_l$. Because f_l is not identically zero there must be some k with $u_k = \bar{f}_l$. This shows that *every* eigensection of Δ_0 is the limit of (a subsequence of) $\{f_{j,k}\}$ and the corresponding eigenvalue $\bar{\lambda_k}$ is the limit of the corresponding eigenvalues.

5.3 Correct Multiplicities

We argue here by contradiction. Let λ be an eigenvalue for Δ_0 with k dimensional eigenspace spanned by $u_1, \ldots, u_k$. Assume λ occurs as an accumulation point of multiplicity less than k, without loss of generality assume multiplicity $k-1$. However, preceding arguments imply the existence of a subsequence $\{f_{k,j}\}$ of Δ_j with $f_{k,j} \to u_k$, which shows that λ is achieved as an accumulation point of multiplicity at least k. Conversely, assume λ occurs as an accumulation point of multiplicity $k+1$. By preceding orthogonality argument, the limit section u_{k+1} of the converging sequence $f_{k+1,j}$ is orthogonal to $\{u_1, \ldots, u_k\}$. This is a contradiction.

♡

6 Heat Kernels

The heat kernels for each of the geometries in ac convergence are elements of a pseudodifferential heat operator calculus that is defined on the corresponding heat space. For the details in the construction of these heat calculi, kernels and spaces, see [29].

6.1 b-heat kernel

Let (M, g) be a b-manifold with local coordinates $z = (x, y)$ in a neighborhood of ∂M so that near the boundary

$$g = \frac{dx^2}{x^2} + h(x, y).$$

Let (z, z') be coordinates on $M \times M$ and let Δ_b be a geometric Laplacian on M. The b-heat kernel $H(z, z', t)$ is the Schwartz kernel of the fundamental solution of the heat operator $\partial_t + \Delta_b$. The heat kernel is a distributional section that acts on smooth sections of M and satisfies

$$(\partial_t + \Delta_b)H(z, z', t) = 0,\ t > 0,$$

$$H|_{t=0} = \delta(z - z').$$

By self adjointness since we work with the Friedrich's extension of Δ_b,

$$H(z, z', t) = H(z', z, t)^*.$$

For a smooth section u on M,

$$u(z, t) := \int_M \langle u(z'), H(z, z', t)\rangle dz'$$

satisfies

$$(\partial_t + \Delta_b)u(z, t) = 0 \text{ for } t > 0, \quad u(z, 0) = u(z).$$

Physically, $u(z, t)$ describes the heat on M at time $t > 0$ where the initial heat applied to M is given by $u(z)$.

Recall the Euclidean heat kernel,

$$G(z, z', t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|z - z'|^2}{2t}\right).$$

For a compact manifold without boundary, the heat kernel can be constructed locally using the Euclidean heat kernel and geodesic normal coordinates [28]. On the interior of a manifold with boundary (or singularity) the Euclidean heat kernel is also a good model, however, near the boundary (singularity) a different construction is required.

6.1.1 b-heat space

It is convenient to study the heat kernel on a manifold with boundary (or singularity) as an element of a heat operator calculus defined on a corresponding heat space. This space is a manifold with corners constructed from $M \times M \times \mathbb{R}^+$ by blowing up along submanifolds at which the heat kernel may have interesting or singular behavior. For example, the diagonal is always blown up at $t = 0$, since away from the boundary the heat kernel behaves like the

Euclidean heat kernel which is singular along the diagonal at $t = 0$. For the b-heat space we first blow up the codimension 2 corner at the boundary in both copies of M. The *b-heat space* $M^2_{b,h}$ is then,

$$M^2_{b,h} = \left[M^2_b \times \mathbb{R}^+_t; \Delta(M \times M) \times \{t = 0\}, dt\right],$$

where $\Delta(M \times M)$ is the diagonal in $M \times M$ and M^2_b is the b-double space (2). The b-heat space has five boundary faces, two of which result from blowing up. The remaining three boundary faces are at $t = 0$ off the diagonal and at the boundary in each copy of M. More precisely, we have the following.[8]

Face	Geometry of face	Defining function in local coordinates
F_{110}	$SN^+(Y \times Y) \times \mathbb{R}^+$	$\rho_{110} = (x^2 + (x')^2)^{\frac{1}{2}}$
F_{d2}	$PN^+_t(\Delta(M \times M))$	$\rho_{d2} = (\lvert z - z'\rvert^4 + t^2)^{\frac{1}{4}}$
F_{100}	$Y \times (M - \partial M) \times \mathbb{R}^+$	$\rho_{100} = x$
F_{010}	$Y \times (M - \partial M) \times \mathbb{R}^+$	$\rho_{010} = x'$
F_{001}	$(M - \partial M)^2 - \Delta(M \times M)$	$\rho_{001} = t$

Above PN^+_t denotes the inward pointing t parabolic normal bundle while SN^+ denotes the inward pointing spherical normal bundle. Note that the local coordinates x, x', t lift from $M \times M \times \mathbb{R}^+$ to $M^2_{b,h}$ as follows,

$$\beta^*(x) = \rho_{110}\rho_{100},\ \beta^*(x') = \rho_{110}\rho_{010},\ \beta^*(t) = \rho^2_{d2}\rho_{001},$$

so these coordinates are only *local* defining functions.

6.1.2 b-heat calculus

The b-heat calculus consists of distributional section half density kernels on $M^2_+ = M \times M \times \mathbb{R}^+$ which are smooth on the interior and lift to be polyhomogeneous on $M^2_{b,h}$ with specified leading orders at the boundary faces. By constructing the b-heat kernel as an element of the b-heat calculus, it is polyhomogenous on $M^2_{b,h.}$ and we know the leading order terms. Consequently, the b-heat kernel is polyhomogeneous at the boundaries and corners of M^2_+ with specified leading orders. Once the calculus is defined and the composition rule is proven, construction of the heat kernel as an element of the heat calculus is similar to solving an ordinary differential equation using Taylor series. The following definition is from [26].

Definition 10. *For any $k \in \mathbb{R}$ and index set E_{110}, A is an element of the b-heat calculus, $\Psi^{E_{110},k}_{b,H}$ if the following hold.*

1. $A \in \mathcal{A}^{-\frac{1}{2}+E_{110}}_{phg}(F_{110})$.

[8] The subscript "d" indicates a face created by blowing up along the diagonal, so for example F_{d2} is the face created by blowing up along the diagonal parabolically in the t direction.

2. A vanishes to infinite order at F_{001}, F_{100}, and F_{010}.

3. $A \in \rho_{d2}^{-\frac{n+3}{2}-k}\mathcal{C}^\infty(F_{d2})$.

Because the heat calculus is defined with half densities, the normalizing factors at F_{110} and F_{d2} simplify the composition rule. An element A of the b-heat calculus is the Schwartz kernel of an operator acting on a smooth half density section f of M by

$$Af(z,t) = \int_M \langle A(z,z',t), f(z')\rangle dz'.$$

Furthermore, A acts by convolution in the t variable so for a smooth half density section f of $M \times \mathbb{R}_t^+$,

$$Af(z,t) = \int_0^t \int_M \langle A(z,z',t-s), f(z',s)\rangle dz' ds.$$

Two elements of the b-heat calculus compose as follows.

Technical Theorem 1. *Let $A \in \Psi_{b,H}^{k_a,\mathcal{A}}$ and let $B \in \Psi_{b,H}^{k_b,\mathcal{B}}$. Then the composition, $A \circ B$ is an element of $\Psi_{b,H}^{k_a+k_b,\mathcal{A}+\mathcal{B}}$.*

The proof of this composition rule is in [26].[9]

6.1.3 Construction of the b-heat kernel

First we construct a model heat kernel H_1 as an element of the b-heat calculus that solves the heat equation up to an error vanishing to positive order at the boundary faces of $M_{b,h}^2$. On the interior of $M_{b,h}^2$ restricting to a coordinate patch with coordinates (z,z',t), we locally define

$$H_1(z,z',t) := (4\pi t)^{-n/2} e^{(|z-z'|_g)^2/2t},$$

where $|z-z'|_g$ is the distance from z to z' with respect to the metric g. As $t \to 0$ away from the diagonal this construction immediately implies infinite order vanishing at F_{001}. At F_{d2} we solve exactly: for each $p \in M$ and for each point $z \in F_{d2}$ in the fiber over $(p,p,0)$ the heat kernel at that point is determined by the coefficients of the metric (and its derivatives) at p.

The *normal operator* of $\partial_t + \Delta_b$ is the restriction to F_{110} of the lift of $\partial_t + \Delta_b$ to $M_{b,h}^2$. H_1 is defined at F_{110} to be the kernel of a first order parametrix of this normal operator and is smooth at this face. At F_{100}, F_{010}, and F_{001} the model kernel vanishes to infinite order. As constructed, H_1 satisfies

$$(\partial_t + \Delta)H_1 = K_1, \quad H_1 \in \Psi_{b,H}^{-2,0}$$

[9]Note that in order for the composition to be *defined* A and B must satisfy certain compatibility conditions. In our all our applications these conditions are a priori satisfied.

where K_1 now vanishes to positive order at the boundary faces of $M^2_{b,h}$. Then, define

$$H_2 = H_1 - H_1 * K_1,$$

where now the error term

$$K_2 = (\partial_t + \Delta)H_2$$

vanishes to one order higher on each of the boundary faces of $M^2_{b,h}$ by the composition rule. This construction is iterated and Borel summation (see [30]) gives $H_\infty \in \Psi^{-2,0}_{b,H}$ with $H_\infty - H_N = O(t^{N-\frac{n+3}{2}})$, for $N > 0$, so that

$$(\partial_t + \Delta)H_\infty = K,$$

where K vanishes to infinite order on the boundary faces of $M^2_{b,h}$ so we may push K forward to $M \times M \times \mathbb{R}^+$. We solve away the residual error term using the action of elements of the b-heat calculus as t-convolution operators. As a t-convolution operator, the heat kernel is the identity. Above, K as a t-convolution operator is of the form $K = Id - A$ where A is a Volterra operator and Id is the identity. An operator of this form has an inverse of the same form so defining

$$H := H_\infty(Id - A)^{-1}$$

solves away this residual error term. By construction the leading order behavior of the b-heat kernel is that of the model heat kernel and is summarized below.

Face	Leading order
F_{110}	0; $O(t^{-\frac{1}{2}})$ as $t \to \infty$.
F_{d2}	$-\frac{n+3}{2} - (-2)$
F_{100}	∞
F_{010}	∞
F_{001}	∞

6.2 Conic heat kernel

Let (M_0, g_0) be a compact manifold with isolated conic singularity and let (E_0, ∇_0) be a Hermitian vector bundle over (M_0, g_0). Let Δ_0 be the Friedrich's extension of a geometric Laplacian on (M_0, g_0) and let Y be the smooth $n-1$ dimensional cross section of M_0 so that $\partial M_0 = Y$. The conic heat kernel is constructed analogously to the b-heat kernel.

6.2.1 The conic heat space

This construction comes from [27]. The conic heat space $M^2_{0,h}$ is a manifold with corners obtained from $M_0 \times M_0 \times \mathbb{R}^+ = M^2_{0,+}$ by blowing up along two submanifolds,

$$M^2_{0,h} := \left[[M_0 \times M_0 \times \mathbb{R}^+; \partial M_0 \times \partial M_0 \times \{t = 0\}, dt]; \Delta(M^0_0 \times M^0_0) \times \{t = 0\}, dt\right].$$

The conic heat space has five boundary faces described in the following table in which $z = (x, y)$ and $z' = (x', y')$ are local coordinates in a neighborhood of the singularity in each copy of M_0 so that $x = 0, x' = 0$ define the singularity as well as the boundary of M_0.

Face	Geometry of face	Defining function in local coordinates
F_{112}	$PN_t^+(Y \times Y)$	$\rho_{112} = (x^4 + (x')^4 + t^2)^{\frac{1}{4}}$
F_{d2}	$PN_t^+(\Delta(M_0^0 \times M_0^0))$	$\rho_{d2} = (\lvert z - z' \rvert^4 + t^2)^{\frac{1}{4}}$
F_{100}	$Y \times \mathbb{R}^+$	$\rho_{100} = x$
F_{010}	$\mathbb{R}^+ \times Y$	$\rho_{010} = x'$
F_{001}	$M_0^0 \times M_0^0 - \Delta(M_0^0 \times M_0^0)$	$\rho_{001} = t$

Note that the coordinates x, x', t lift from $M_0 \times M_0 \times \mathbb{R}^+$ to $M_{0,h}^2$ as follows,

$$\beta^*(x) = \rho_{100}\rho_{112},\ \beta^*(x') = \rho_{010}\rho_{112},\ \beta^*(t) = \rho_{112}^2\rho_{d2}^2\rho_{001},$$

so again these are only *local* defining functions.

6.2.2 The conic heat calculus

Let μ be a conic half density on $M_{0,+}^2$; we may assume

$$\mu = (xx')^{\frac{n-1}{2}}\sqrt{\mathrm{d}z\mathrm{d}z'\mathrm{d}t} = \sqrt{\mathrm{d}V_c\mathrm{d}t}.$$

Fix also a smooth, nonvanishing half density, ν, on $M_{0,h}^2$. Elements of the conic heat calculus are distributional section half densities on $M_{0,+}^2$ which are smooth on the interior and lift to be polyhomogeneous on $M_{0,h}^2$.

Definition 11. *Let $k \in \mathbb{R}$ and E_{100} E_{010} E_{112} be index sets. Then $A \in \Psi_{0,H}^{k,E_{100},E_{010},E_{112}}$ if the following hold.*

1. *$A \in \mathcal{A}_{phg}^{E_{100}}$ at F_{100}.*
2. *$A \in \mathcal{A}_{phg}^{E_{010}}$ at F_{010}.*
3. *$A \in \mathcal{A}_{phg}^{E_{112}}$ at F_{112}.*
4. *A vanishes to infinite order at F_{001}.*
5. *$A \in \rho_{d2}^{-\frac{n+3}{2}-k}\mathcal{C}^\infty(F_{d2})$.*

With this normalization the conic heat kernel has order $k = -2$ and the composition rule is the following.

Technical Theorem 2. *Let* $A \in \Psi_{0,h}^{A_{100},A_{010},A_{112},k_a}$*, and* $B \in \Psi_{0,H}^{B_{100},B_{010},B_{112},k_b}$ *with the leading index terms satisfying*

$$\beta_{112} + \alpha_{010} > 0,\ \alpha_{112} + \beta_{100} > 0,\ -k_a > 0,\ -k_b > 0,\ \beta_{100} + \alpha_{010} > -1.$$

Then, the composition $B \circ A$ *is an element of* $\Psi_{0,H}^{A_{100},B_{010},\Gamma_{112},k}$ *with* $\Gamma_{112} = A_{112} + B_{112}$ *and* $k = (k_a + k_b)$*.*

The proof of this theorem is in [29], and is originally due to [27], see also [22], [12].

The conic heat kernel is constructed analogously to the b-heat kernel first using a model heat kernel and then using the composition rule to iteratively solve away the error term. Away from F_{112}, F_{100} and F_{010} the model heat kernel comes from the standard local construction using the Euclidean heat kernel. At F_{112} the model heat kernel comes from an explicit construction of the heat kernel for an exact cone transplanted to F_{112} and extended smoothly to F_{100} and F_{010}. See, for example [3] and [27] or for the case of the scalar Laplacian, [29].

The most interesting behavior of the conic heat kernel is at the front face F_{112}, so we briefly review the construction of the model heat kernel at this face. At F_{112} the coordinates x, x', t are not good. Instead, consider the projective coordinates

$$s = x/x',\ s' = x',\ \tau = t/(x')^2.$$

In these coordinates the heat operator

$$\partial_t + \Delta \to (s')^{-2}(\partial_\tau + (\partial_s)^2 + s^{-2}\Delta_h),$$

where Δ_h is the Laplacian for (Y, h). Then, we see that the model heat kernel at F_{112} should be

$$(\rho_{112})^2 H_0(s, s', y, y', \tau),$$

where H_0 is the heat kernel for the *exact* cone over (Y, h). Using the scaling properties of the heat kernel for an exact cone, we see this is equivalent to

$$(\rho_{112})^{2-n} H_0(s, 1, y, y', \tau),$$

which is well defined away from F_{010}, and by the symmetry in space variables we may then use this to define the model heat kernel on all of F_{112}. By the properties of the heat kernel for the exact cone, (see [3]) this implies the existence of a full polyhomogeneous expansion at the side faces F_{100}, F_{010}.

6.3 Ac scattering heat kernel

A summary of the ac scattering heat kernel, space and calculus is given here; for the details, see the appendix. Let $\bar{Z}$ be a compactified ac scattering space with boundary defined by $\{x = 0\}$ and local coordinates (x, y) near the boundary. Let Δ_z be the Friedrich's extension of a geometric Laplacian on Z.

6.3.1 The ac scattering heat space

First, we construct the ac scattering double space,

$$\bar{Z}^2_{sc} := \left[[\bar{Z} \times \bar{Z}; \partial\bar{Z} \times \partial\bar{Z}]; \Delta(Y \times Y) \cap F_{110}\right]$$

where F_{110} is the face created by the first blowup. This construction comes from [14]. Then, the ac scattering heat space is

$$\bar{Z}^2_{sc,h} = \left[\bar{Z}^2_{sc} \times \mathbb{R}^+; \Delta(Z \times Z) \times \{t = 0\}, dt\right].$$

The ac scattering heat space has six boundary faces described in the following table.

Face	Geometry of face	Defining function in local coordinates
F_{220}	$N^+(\Delta(Y \times Y)) \times \mathbb{R}^+$	$\rho_{220} = (x^2 + (x')^2 + \lvert y - y'\rvert^2)^{\frac{1}{2}}$
F_{220}	$N^+((Y \times Y) - \Delta(Y \times Y)) \times \mathbb{R}^+$	$\rho_{110} = (x^2 + (x')^2)^{\frac{1}{2}}$
F_{100}	$Z \times Y \times \mathbb{R}^+$	$\rho_{100} = x$
F_{010}	$Y \times Z \times \mathbb{R}^+$	$\rho_{010} = x'$
F_{d2}	$PN_t^+(\Delta(Z \times Z))$	$\rho_{d2} = (\lvert z - z'\rvert^4 + t^2)^{\frac{1}{2}}$
F_{001}	$(Z \times Z) - \Delta(Z \times Z)$	$\rho_{001} = t$

6.3.2 Ac scattering heat calculus

Elements of the ac scattering heat calculus are distributional section half densities of Z^2_+ which are smooth on the interior and lift to be polyhomogeneous on $\bar{Z}^2_{sc,h}$. Let μ be a smooth, non-vanishing half density on $\bar{Z}^2_+$ and let ν be a smooth, non-vanishing half density on $\bar{Z}^2_{sc,h}$.

Definition 12. *For any $k \in \mathbb{R}$ and index sets E_{110}, E_{220}, $A \in \Psi^{E_{110}, E_{220}, k}_{sc,H}$ if the following hold.*

1. *$A \in \mathcal{A}^{-\frac{1}{2}+E_{110}}_{phg}$ at F_{110}.*
2. *$A \in \mathcal{A}^{-\frac{n+2}{2}+E_{220}}_{phg}$ at F_{220}.*
3. *A vanishes to infinite order at F_{001}, F_{100}, and F_{010}.*
4. *$A \in \rho_{d2}^{-\frac{n+3}{2}-k}\mathcal{C}^\infty(F_{d2})$.*

Two elements of the ac scattering heat calculus compose as follows.

Technical Theorem 3. *Let $A \in \Psi^{A_{110}, A_{220}, k_a}_{sc,H}$, and $B \in \Psi^{B_{110}, B_{220}, k_b}_{sc,H}$.*

Then, the composition $B \circ A$ is an element of $\Psi^{A_{110}+B_{110}, A_{220}+B_{220}, k_a+k_b}_{sc,H}$.

The ac scattering heat kernel is constructed analogously to the b and conic heat kernels. The model heat kernel in this case is the lift of the Euclidean heat kernel to $\bar{Z}^2_{sc,h}$ and by construction the leading orders of the ac scattering heat kernel are that of the model kernel at the boundary faces of $\bar{Z}^2_{sc,h}$. This is stated in the following theorem.

Technical Theorem 4. *Let (Z, g_z) be an asymptotically conic manifold with cross section (Y, h) at infinity. Let (E, ∇) be a Hermitian vector bundle over (Z, g_z) which induces a compatible bundle over (Y, h). Let Δ be a geometric Laplacian on (Z, g_z) associated to the bundle (E, ∇). Then there exists $H \in \Psi^{E_{110}, E_{220}, -2}_{sc,H}$ satisfying:*

$$(\partial_t + \Delta)H(z, z', t) = 0,\ t > 0,$$
$$H(z, z', 0) = \delta(z - z').$$

Moreover, H vanishes to infinite order at F_{110} and is smooth up to F_{220}.

The proof of this theorem is in the appendix.

7 Heat Kernel Convergence

The interaction of the heat kernels will be studied on the asymptotically conic convergence (acc) heat space.

7.1 The acc heat space

The acc heat space construction is similar to the heat space constructions of section 6 and the double space construction in section 3. First, let

$$\mathcal{H}_0 := \{\epsilon = \epsilon'\} \subset \mathcal{S} \times \mathcal{S}.$$

Next, let

$$\mathcal{H}_1 := [\mathcal{H}_0 \times \mathbb{R}^+_t; Y \times Y \times \{t = 0\}, dt].$$

This blowup must be done first to create the F_{112} face in the conic heat space. The scalar variables on $\mathcal{S} \times \mathcal{S} \times \mathbb{R}^+_t$ are (x, r, x', r', t),[10] so by our notation the face created by this blowup is $F_{1111,2}$. Let

$$\mathcal{H}_2 := [\mathcal{H}_2; Z \times Z \times \{t = 0\}, dt].$$

This blowup is not obvious: in the calculations to follow in which we lift the heat operator to the acc heat space and calculate its behavior as $\epsilon \to 0$, we see a b heat operator with rescaled time variable. This blowup is necessary to create a compactified b-heat space at

[10] Note that these variables are not independent; they are related by $xr = x'r' = \epsilon$.

$t, \epsilon = 0$. The resulting face is $F_{1010,2}$. Finally, the acc heat space results from blowing up the lift of the diagonal in $\mathcal{S} \times \mathcal{S}$ at the lift of $\{t = 0\}$ away from $F_{1111,2}$.

$$\mathcal{H} := [\mathcal{H}_2; \beta^*(\Delta(\mathcal{S} \times \mathcal{S}) - (Y \times Y)) \cap F_{0000,1}].$$

The face created by this last blowup is F_{d2}.

The $\epsilon = 0$ boundary faces of $\mathcal{H}$ are summarized below.

$\mathcal{S} \times \mathcal{S} \times \mathbb{R}^+_t$ corner	$\mathcal{H}$ face	geometry
$x = 0, x' = 0, r = 0, r' = 0, t = 0$	$F_{1111,2}$	$PN^+(Y \times Y)$
$x = 0, x' = 0, t = 0,$	$F_{1010,2}$	$\left[\bar{Z} \times \bar{Z} \times \mathbb{R}^+_\tau; Y \times Y\right]; \Delta(Z \times Z) \times \{0\}, d\tau\right]$
$x = 0, x' = 0,$	F_{1010}	$[Z \times Z \times \mathbb{R}^+ - \{t = 0\}]$
$r = 0, r' = 0$	F_{0101}	$\left[[M_0 \times M_0 \times \mathbb{R}^+; Y \times Y \times \{0\}, dt]; \Delta(M_0^0 \times M_0^0) \times \{0\}, dt\right]$
$x = 0, r' = 0$	F_{1001}	$[\bar{Z} \times M_0 \times \mathbb{R}^+; Y \times Y \times \{0\}, dt]$
$r = 0, x' = 0$	F_{0110}	$[M_0 \times \bar{Z} \times \mathbb{R}^+; Y \times Y \times \{0\}, dt]$
$\Delta(\mathcal{S} \times \mathcal{S}) \times \{t = 0\}$	F_{d2}	$PN^+(\Delta(\mathcal{S} \times \mathcal{S}) - Y \times Y)$
$\{t = 0\}$	$F_{0000,1}$	$(\{\epsilon = \epsilon'\} \subset \mathcal{S} \times \mathcal{S}) - (\Delta(\mathcal{S} \times \mathcal{S}) \cup Y \times Y)$

7.1.1 Acc half density calculations

We calculate the lift to $\mathcal{H}$ of $dVg_\epsilon dVg'_\epsilon dt d\epsilon$. The Jacobian determinant factors which result from blowing up are

$$(\rho_{1111.2})^4 (\rho_{1010,2})^2 (\rho_{d2})^{n+2}.$$

[11] Next, we calculate the lift of the variables x, x', r, r' to $\mathcal{H}$,

$$\beta^*(x) = \rho_{1010}\rho_{1010,2}\rho_{1111,2}\rho_{1001},$$

$$\beta^*(x') = \rho_{1010}\rho_{1010,2}\rho_{1111,2}\rho_{0110},$$

$$\beta^*(r) = \rho_{1111,2}\rho_{0101}\rho_{0110},$$

$$\beta^*(r') = \rho_{1111,2}\rho_{0101}\rho_{1001}.$$

We calculate the volume form dVg_ϵ in a neighborhood of the faces of $\mathcal{H}$ at $\epsilon = 0$. At F_{0101}, $dVg_\epsilon \sim dV_0$, the conic density, consequently

$$dVg_\epsilon dVg'_\epsilon \to (\rho_{1111,2}\rho_{1010}\rho_{1010,2})^{2n-2} (\rho_{1001}\rho_{0110})^{n-1} \mu.$$

At $F_{1010}, F_{1010,2}$, $dVg_\epsilon \sim \epsilon^2 dV_z$, the ac density, consequently

$$dVg_\epsilon dVg'_\epsilon \to (\rho_{1111,2}\rho_{1010,2}\rho_{1010})^{2n} (\rho_{0101})^{-2} (\rho_{1001}\rho_{0110})^{n-1} \mu,$$

where μ is a smooth nonvanishing spatial density on $\mathcal{H}$.[12]

[11] The recipe for these exponents is: (codimension of space variables -1) + (codimension of parabolic variables $*2$).

[12] We have used that $dV_0 \sim x^{n-1} dx dy$ and $dV_z \sim \frac{\epsilon^n}{r^{n+1}} dr dy$ and that $\epsilon = xr = x'r'$.

Then, we arrive at the following half density calculation: at F_{0101},

$$\beta^*(\sqrt{dVg_\epsilon dVg'_\epsilon dtd\epsilon}) \sim (\rho_{1111,2})^{n+1}(\rho_{1010,2})^n(\rho_{1010})^{n-1}(\rho_{1001}\rho_{0110})^{(n-1)/2}\sqrt{\nu},$$

where ν is a smooth nonvanishing density on $\mathcal{H}$. At F_{1010} and $F_{1010,2}$,

$$\beta^*(\sqrt{dVg_\epsilon dVg'_\epsilon dtd\epsilon}) \sim (\rho_{1111,2})^{n+1}(\rho_{1010,2})^{n+1}(\rho_{1010})^n(\rho_{0101})^{-2}(\rho_{1001}\rho_{0110})^{(n-1)/2}\sqrt{\nu}.$$

Note that these are not the same! In constructing the acc model heat kernel, we will correct the discrepancy so that the model heat kernel is normalized as an element of the acc heat calculus.

7.2 The acc heat calculus

The acc heat calculus is a parameter (ϵ) dependendent operator calculus incorporating the smooth, conic, and b-heat calculi.[13]

Definition 13. *The* asymptotically conic convergence heat calculus of order k, *written* $\Psi^{k,E_{0101},E_{1010,2},E_{1111,2}}_{acc,H}$ *consists of kernels A such that the following hold.*

1. *For each $\epsilon > 0$, A restricts to an element of $\Psi^k_{\epsilon,H}$, the smooth compact heat calculus of order k.*
2. *In a neighborhood of $F_{1010,2}$, A has an asymptotic expansion in $\rho_{1010,2}$ with index set $E_{1010,2}$ and coefficients in the b-heat calculus of order k. Such an expansion is of the form*

$$A \sim \sum_{j\geq 1}\sum_{0\leq p_0\leq p\leq p_j}(\rho_{1010,2})^{\alpha_j}(\log\rho_{1010,2})^p A_{j,l},$$

   with $A_{j,l} \in \Psi^{k,E^j_{110}}_{b,H}$. Above, if for some j, $p_j = 0$, then there are no log *terms.*
3. *In a neighborhood of F_{0101}, A has an asymptotic expansion in ρ_{0101} with index set E_{0101} and coefficients are elements of the conic heat calculus of order k.*
4. *In a neighborhood of $F_{1111,2}$, A has an asymptotic expansion in $\rho_{1111,2}$ with index set $E_{1111,2}$ so that the coefficients in the conic heat calculus of order k for the exact cone over Y.*
5. *A vanishes to infinite order at $F_{1001}, F_{1010}, F_{0110}$.*

[13]The ac scattering heat calculus was expected to arise in the acc heat space and heat calculus, but after calculating the behavior of the heat kernel as $\epsilon \to 0$, it is clear that the ac scattering heat space is not needed for the acc heat space and the ac scattering heat kernel does not appear.

The composition rule is not required for the proof of our main theorem, but we expect it to follow from the composition rules for the smooth, b, and conic heat calculi, together with the result for combining polyhomogeneous index sets as in [23]. In appendix B we construct the acc triple heat space, the key technical tool for proving the composition rule.

Theorem 2. *Let (M_0, g_0) be a compact Riemannian n-manifold with isolated conic singularity, and let (Z, g_z) be an asymptotically conic space, with $n \geq 2$. Assume (M, g_ϵ) converges asymptotically conically to (M_0, g_0). Let (E_0, ∇_0) and (E_z, ∇_z) be Hermitian vector bundles over (M_0, g_0) and (Z, g_z), respectively, so that each of these bundles in a neighborhood of the boundary is the pullback from a bundle over the cross section (Y, h). Let Δ_0, Δ_z be the corresponding Friedrich's extensions of geometric Laplacians, and let Δ_ϵ be the induced geometric Laplacian on (M, g_ϵ). Then the associated heat kernels H_ϵ have a full polyhomogeneous expansion as $\epsilon \to 0$ on the asymptotically conic convergence (acc) heat space with the following leading terms:*

- *At the conic front face, F_{0101}, $H(z, z', t, \epsilon) \to H_0(z, z', t)$, the heat kernel for (M_0, g_0).*
- *At the rescaled b front face, $F_{1010,2}$, $H(z, z', t, \epsilon) \to (\rho_{1010,2})H_b(\tau)$, the b heat kernel with rescaled time variable τ.*
- *At the exact conic front face, $F_{1111,2}$, $H(z, z', t, \epsilon) \to (\rho_{1111,2})^2 H_0(\tau)$, the heat kernel for the exact cone with rescaled time variable τ.*
- *At the side faces F_{1001}, F_{0110} and the residual b face F_{1010}, the heat kernel vanishes to infinite order.*

This convergence is uniform in ϵ for all time and moreover, the error term is bounded by $C_N \epsilon t^N$ as $t \to 0$, for any $N \in \mathbb{N}_0$.

7.3 Proof

This proof is modeled after the parametrix construction of [26]. First, we lift the operator $\partial_t + \Delta_\epsilon$ to $\mathcal{H}$ and construct the *acc model heat kernel* as an element of the acc heat calculus which solves the leading terms of the lifted operator. We estimate the error term of this solution kernel, use the b-heat calculus, introduce the acc conic triple heat space, use the conic heat calculus and finally solve away the residual error term.

7.3.1 Lifted heat operator

We must carefully choose good coordinates in a neighborhood of each of the front faces $F_{0101}, F_{1010,2}$ and $F_{1111,2}$ and calculate the leading term of $\partial_t + \Delta_\epsilon$ as $\epsilon \to 0$. In a neighborhood of F_{0101}, the g_ϵ is smoothly approaching the conic metric, so

$$\partial_t + \Delta_\epsilon \to \partial_t + \Delta_0.$$

In a neighborhood of $F_{1010,2}$, the metric $g_\epsilon \to \epsilon^{-2} g_z$, so

$$\partial_t + \Delta_\epsilon \to \partial_t + \epsilon^{-2}\Delta_z = (x)^{-2}(\partial_\tau + \Delta_b),$$

where we have used $xr = \epsilon$ and the relation between the ac scattering Laplacian and the rescaled b Laplacian $\Delta_z = r^2\Delta_b$. Note that x lifts to define $F_{1010}, F_{1010,2}$, and $F_{1111,2}$. This indicates that the leading part of $\partial_t + \Delta_\epsilon$ at $F_{1010,2}$ is

$$(\rho_{1010,2}\rho_{1010}\rho_{1111,2})^{-2}(\partial_\tau + \Delta_b).$$

In a neighborhood of $F_{1111,2}$, the scalar variables (x, x', r, r', t) are not good coordinates since they all vanish. Better coordinates are the projective $(s, s', \sigma, \sigma', \tau)$, where

$$s := x/x',\ s' = x',\ \sigma = r/x',\ \sigma' = r'/x',\ \tau = t/(x')^2.$$

Note that

$$(r\partial_r) = (\sigma\partial_\sigma), \quad xr = \epsilon \implies \frac{r^2}{\epsilon^2} = x^{-2}.$$

At $F_{1111,2}$ we see both the conic metric g_0 near the singularity and the rescaled ac scattering metric $\epsilon^2 g_z$ near the boundary. Using the projective coordinates around the conic singularity we compute

$$\partial_t + \Delta_\epsilon \to (s')^{-2}(\partial_\tau + (\partial_s)^2 + (s)^{-2}(\Delta_h) = (ss')^{-2}(\partial_{\tilde\tau} + (s\partial_s)^2 + \Delta_h),$$

where Δ_h is the Laplacian on Y for $h = h(x = 0)$ and $\tilde\tau = \frac{t}{(ss')^2}$. Using the projective coordinates near the boundary of Z we compute

$$\partial_t+\Delta_\epsilon \to (s')^{-2}(\partial_\tau+(s)^{-2}((\sigma\partial_\sigma)^2+\Delta_h) = (ss')^{-2}(\partial_{\tilde\tau}+(\sigma\partial_\sigma)^2+\Delta_h) = (ss')^{-2}(\partial_{\tilde\tau}+(s\partial_s)^2+\Delta_h),$$

where in the last equality we've used $xr = \epsilon = x'r'$, which gives $s\partial_s = \sigma\partial_\sigma$. Since $(ss') = x$ lifts to $\mathcal{H}$ to define $F_{1111,2}, F_{1010}$ and $F_{1010,2}$ as does s', this indicates that the leading term of $\partial_t + \Delta_\epsilon$ at $F_{1111,2}$ is

$$(\rho_{1111,2}\rho_{1010}\rho_{1010,2})^{-2}(\partial_\tau + \Delta_{0,s}),$$

where $\Delta_{0,s}$ is the Laplacian for the exact cone over (Y, h).

These calculations together with the half density calculation tell us how to define the acc model heat kernel as a parametrix for $\partial_t + \Delta_\epsilon$ as $\epsilon \to 0$.

7.3.2 Acc model heat kernel, H_1

- At F_{0101}, let $H_1(z, z', t, \epsilon) \sim H_0$, the heat kernel for (M_0, g_0).
- At $F_{0101,2}$, let $H_1(z, z', t, \epsilon) \sim (\rho_{1010,2}\rho_{1010})(\rho_{0101})^2(\rho_{1111,2})^2 H_b(\tau)$, the b heat kernel with rescaled time variable.[14]

[14] At $F_{1010,2}$ the density discrepancy requires a factor of $(\rho_{1010,2}\rho_{1010})^{-1}(\rho_{0101})^2$, and the operator calculation requires a factor of $(\rho_{1010}\rho_{1010,2}\rho_{1111,2})^2$. To correct both, we then need a factor of $(\rho_{1010,2}\rho_{1010})(\rho_{0101})^2(\rho_{1111,2})^2$.

- At $F_{1111,2}$, let $H_1(z,z',t,\epsilon) \sim (\rho_{1111,2})^2(\rho_{0101})^2 H_0(\tau)$, where $H_0(\tau)$ is the heat kernel for the exact cone over (Y,h) with rescaled time.

Note that we suppress the half density factor

$$(\rho_{1111,2})^{n+1}(\rho_{1010,2})^n(\rho_{1010})^{n-1}(\rho_{1001}\rho_{0110})^{(n-1)/2}(\rho_{d2})^{(n+2)/2}\sqrt{\nu}$$

where ν is a smooth nonvanishing density on $\mathcal{H}$. We define $H_1(z,z',t,\epsilon)$ to vanish to infinite order at $F_{1010}, F_{1001}, F_{0110}$, and define $H_1(z,z',t,\epsilon)$ to be the heat kernel for $\partial_t + \Delta_\epsilon$ for $\epsilon > 0$. This is consistent with the above, since the b heat kernel vanishes to infinite order at both side faces and as $t \to \infty$ away from the diagonal and front face. The b heat kernel vanishes to positive order as $t \to \infty$ along the diagonal and at the front face as well. The above definition is consistent along the corners of $\mathcal{H}$, as can be verified by calculation.

7.3.3 Acc model heat kernel construction along diagonal at $t = 0$

In a neighborhood of the faces diffeomorphic to $PN^+(\Delta)$ in $\mathcal{H}$ we carry out a local construction as in [26] chapter 7. Let X be a manifold; since this construction is the same for $X = M$, $X = M_0$ and $X = \bar{Z}$ we use X to simplify notation. Let F_X denote the $PN^+(\Delta(X))$ face in $\mathcal{H}$, where $\Delta(X)$ is the diagonal in $X \times X$. Let $N(\partial_t + \Delta_X)$ be the restriction to F_X of the lift of $(\partial_t + \Delta_X)$ to $\mathcal{H}$, where Δ_X is our geometric Laplacian on X. With the heat calculus normalization at F_X, an element A of the acc heat calculus of order $-k$ restricts to F_X as follows

$$N(A) = t^{(k+n+2)/2} A|_{F_X}.$$

As in [26] we observe that F_X is naturally diffeomorphic to a radial compactification of the tangent space of X, with each fiber of F_X over $(x,x,0)$ diffeomorphic to the tangent space at x, $fiber(x) \cong T_xX$. Then, from [26] 7.15,

$$N(t(\partial_t + \Delta_X)A) = [\sigma(\Delta_X) - \frac{1}{2}(R+n+k+2)]N(A), \tag{5}$$

where R is the radial vector field on the fibers of TX. Note that if G_0 satisfies

$$t(\partial_t + \Delta_X)G_0 = O(t^\infty) \text{ as } t \to 0, \quad G_0|t=0 = \delta(x-x'),$$

then G_0 also satisfies

$$(\partial_t + \Delta_X)G_0 = O(t^\infty) \text{ as } t \to 0, \quad G_0|t=0 = \delta(x-x'). \tag{6}$$

So, we may work with $t(\partial_t + \Delta_X)$ as in [26]. Our initial parametrix G_0 will have order $k = -2$ at F_X. Then, from 5 we have the following equation for G_0

$$[\sigma(\Delta_X) - \frac{1}{2}(R+n)]N(G_0) = 0.$$

From [26] 7.13, in order for G_0 to satisfy the initial condition it must satisfy,

$$\int_{\text{fiber}} N(G_0) = 1.$$

Since these conditions are fiber-by-fiber we introduce local coordinates so that

$$\sigma(\Delta_X) = D_1^2 + \ldots + D_n^2 \text{ on } T_xX.$$

Then we have

$$\left[D_1^2 + \ldots + D_n^2 - \frac{1}{2}(R+n)\right] N(G_0) = 0, \tag{7}$$

so

$$N(G_0) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|X|_x^2}{4}\right) \tag{8}$$

is the desired solution, where X is a projective local coordinate on F_x, $X = \frac{x-x'}{t^{1/2}}$ (see [26] (7.36)), and $|*|_x$ is the Riemannian norm on TX induced by the metric at x. To see that this is the desired solution, consider the Fourier transform of (7) with $u = N(G_0)|_{T_xX}$,

$$(\xi\partial_\xi + 2|\xi|^2)\hat{u} = 0, \quad \hat{u}(0) = 1.$$

Then by standard results in ordinary differential equations, the expression in (8) is the unique decaying solution.

Now we may iterate this to solve up to higher order. Assume we have found $G_0, \ldots, G_k$ satisfying

$$t(\partial_t + \Delta_X)G_j = R_j,$$

where R_j is of order $-3-j$ at F_X. To find $G_{k+1} = G_k - T_k$, we wish to solve

$$t(\partial_t + \Delta_X)T_k = R_k + R_{k+1},$$

where we have already found R_k of order $-3-k$, and R_{k+1} will be of order $-4-k$. Lifting to TX this becomes

$$\left[\sigma(\Delta_X) - \frac{1}{2}(R+n-j-1)\right] N(T_k) = N(R_k),$$

which we may again solve via Fourier transform. Letting $u = N(T_k)$ and $f = N(R_k)$ we find

$$\hat{u}(\xi) = \int_0^1 \exp((r-1)|\xi|^2)\hat{f}(r\xi)r^{k+1}dr$$

is the desired solution. This completes the inductive construction for all k. Now the successive $T_j = G_{j+1} - G_j$ give a formal power series at F_X which can be summed by Borel's Lemma so that G is order -2 at F_X and satisfies (6). We then set the acc model heat kernel $H_1 = G$ in a neighborhood of F_X.

7.3.4 Acc model heat kernel construction off diagonal at $t = 0$

Consider $F_{1010,2} \cong Z^2_{b,h,\tau}$. This face has the following geometry.

Boundary Face	Geometry of Face	Arising from
F_{bd2}	$PN^+(\Delta(Z \times Z))$	parabolic blowup of diagonal at $\tau = 0$
F_{b110}	$SN^+(Y \times Y) \times \mathbb{R}^+$	blowup of $Y \times Y$ for all τ
F_{b100}	$Y \times Z \times \mathbb{R}^+$	boundary in first copy of Z
F_{b010}	$Z \times Y \times \mathbb{R}^+$	boundary in second copy of Z
F_{b001}	$(Z \times Z) - \Delta(Z \times Z)$	$\tau = 0$ away from diagonal

Since at this face H_1 is asymptotic to $H_b(z, z', \tau)$, at the boundary face where τ vanishes away from the diagonal, H_1 vanishes to infinite order. Recall H_b vanishes to infinite order at the side faces F_{b100} and F_{b010}. At the diagonal face F_{bd2} we've solved H_1 up to error vanishing to infinite order in t. So, we have at this point an approximation H_1 whose error vanishes to infinite order on the interior of F_{1010} and at all boundary faces except F_{b110}. The indicial operator for $\Delta_{b,\sigma}$ at F_{b110} is

$$(\sigma\partial_\sigma)^2 + \Delta_y$$

on $\mathbb{R}^+_\sigma \times Y$. We would like to solve

$$(\partial_\tau + \Delta_b)|_{F_{b110}} u = -K|_{F_{b110}},$$

so that u is polyhomogeneous on $\widetilde{Z}^2_{b,h}$; then $H_1 + u$ would vanish identically at F_{b110}. Since H_1 is polyhomogeneous on F_{1010} and smooth up to these boundary faces, the error term K is also. Expanding K, where we use simply ρ for the projective defining function for F_{b110},

$$K \sim \sum_{j \geq 0} (\rho)^j k_j, \quad k_j \in \mathcal{C}^\infty_0,$$

and expanding the desired solution u,

$$u \sim \sum_{j \geq 0} (\rho)^j u_j$$

we may use either separation of variables expanding in eigenfunctions of Δ_y or the Mellin transform (see [26]) to find u_0 satisfying

$$(\partial_\tau + I(\Delta_b))u_0 = -k_0,$$

with u_0 vanishing to infinite order as $\sigma \to 0, \infty$: at the side faces F_{b100}, F_{b010}. Since $\Delta_b - I(\Delta_b) = (\rho)(L_1)$, where L_1 is also a b-differential operator, we may now iteratively solve for $u_1, u_2, \ldots,$ to solve the equation to increasingly higher order. Recall that in the projective coordinates near this face, σ' defines F_{b110} and since the operator does not differentiate with respect to $\sigma' = r'$ the defining function commutes past the operator. Using Borel summation we construct u so that

$$(\partial_\tau + \Delta_b)u = -K + K_2$$

where K_2 vanishes to infinite order at F_{b110}. Using a smooth cutoff function χ supported in a neighborhood of these faces, the second approximation $H_2 = H_1 + \chi u$ now satisfies

$$(\partial_\tau + \Delta_b)H_2|_{F_{11}} = K_2$$

where K_2 vanishes to infinite order on both the interior and all boundary faces of F_{1010}.

A similar exact construction applies to $F_{1111,2}$, since this face is the blown down heat space for the *exact* cone over (Y, h).

7.3.5 Error term approximation

For each $\epsilon > 0$, let $E(z, z', t, \epsilon) = H_\epsilon(z, z', t) - H_1(z, z', t, \epsilon)$. Let K be defined for each $\epsilon > 0$ by

$$(\partial_t + \Delta_\epsilon)E(z, z', t, \epsilon) = K(z, z', t, \epsilon).$$

By construction of H_1, $K = O(\epsilon t^\infty)$ as $\epsilon, t \to 0$, so for any$N \in \mathbb{N}$, there is $C > 0$ such that for any $(z, z') \in M \times M$,

$$|K(z, z', t, \epsilon)| < C\epsilon t^N.$$

Moreover, K has a polyhomogeneous expansion down to $\epsilon = 0$.

For each $\epsilon > 0$, E is smooth on $\mathcal{H}$ for $t > 0$ by parabolic regularity applied for each $\epsilon > 0$ since K is $O(t^\infty)$. By construction E is smooth down to $t = 0$, so $E(z, z', t, \epsilon)$ is smooth on the blown down space, $M \times M \times \mathbb{R}^+ \times (0, \delta]_\epsilon$. The following maximum principle argument on $M \times [0, T]_t$ shows that E is also $O(\epsilon t^\infty)$ as $\epsilon, t \to 0$ in the same sense as K.

Fix $\epsilon > 0$, $z' \in M$. Since $K = O(\epsilon t^\infty)$, fix $C > 1$ and $\mathbb{N} \ni N >> 1$ such that $|K(z, z', t, \epsilon)|^2 \leq C\epsilon^2 t^{2N}$ for all $z \in M$. Let $u(z, t) = |E(z, z', t, \epsilon)|^2$. Let Δ be the scalar Laplacian for (M, g_ϵ). Then u satisfies

$$(\partial_t + \Delta)u = 2\langle(\partial_t + \nabla^*\nabla)E, E\rangle - |\nabla E|^2 \;=\; 2\langle K - \mathcal{R}E, E\rangle - |\nabla E|^2$$

$$\leq 2\langle K, E\rangle \leq 2|K||E| \leq |K|^2 + |E|^2 = |K|^2 + u.$$

Above we have used the positivity of $\mathcal{R}$ and the compatibility of the bundle connection with the metric. Now, let $\tilde{u} = e^{-t}u$. Then $\tilde{u}$ satisfies

$$(\partial_t + \Delta)\tilde{u} \leq e^{-t}|K|^2 \leq C\epsilon^2 t^{2N}.$$

Let $w = \tilde{u} - C\epsilon^2 t^{2N+1}$. Since E and hence u and $\tilde{u}$ vanish at $t = 0$, $w|_{t=0} = 0$ and w satisfies

$$(\partial_t + \Delta)w \leq C\epsilon^2 t^{2N} - C(2N + 1)\epsilon^2 t^{2N} < 0.$$

Fix $T > 0$ and consider w on $M \times [0,T]_t$. If w has a local maximum for $z \in M$ and $t \in (0,T)$ then

$$(\partial_t + \Delta)w > 0,$$

and this is a contradiction. If w has a maximum at $t = T$ then $\partial_t w \geq 0$ and

$$(\partial_t + \Delta)w > 0,$$

which is again a contradiction. Therefore, the maximum of w occurs at $t = 0$ and so

$$w \leq C\epsilon^2 t^{2N+1}.$$

This implies

$$u \leq e^T C\epsilon^2 t^{2N}, \quad \text{for } 0 < t \leq T,$$

which in turn implies that $E = O(\epsilon t^N)$ as $\epsilon, t \to 0$, for any $N \in \mathbb{N}$.

7.3.6 Acc conic triple heat space

Since the error now vanishes to infinite order at all boundary faces except F_{0101}, we restrict attention to this face. It is convenient to use the conic heat calculus composition rule, but this requires the conic triple space so we construct a partial acc triple heat space, the acc conic triple heat space, which contains the conic triple heat space so that we may use the conic heat calculus composition rule. The acc conic triple heat space $\mathcal{H}^3_c$ is a submanifold constructed from $\mathcal{S}^3 \times \mathbb{R}^+ \times \mathbb{R}^+$ by eight blowups. Let X, X', X'' denote the three copies of the submanifold X in X^3. Then the blowups are listed in the following table in the order which the blowups are performed together with the name of the face created.

Blowup	Face
$Y \times Y' \times Y'' \times \{t = 0, t' = 0\}, dt, dt'$	F_{11122}
$Y \times Y' \times \{t = 0\}, dt$	F_{11020}
$Y' \times Y'' \times \{t' = 0\}, dt'$	F_{01102}
$Y \times Y'' \times \{t'' = \lvert t - t'\rvert = 0\}, dt''$	F_{10122}
$\Delta(\mathcal{S} \times \mathcal{S}' \times \mathcal{S}'') \times \{t = 0, t' = 0\}, dt, dt'$	F_{d3}
$\Delta(\mathcal{S} \times \mathcal{S}') \times \{t = 0\}, dt$	F_{d20}
$\Delta(\mathcal{S}' \times \mathcal{S}'') \times \{t' = 0\}, dt'$	F_{d02}
$\Delta(\mathcal{S} \times \mathcal{S}'') \times \{t'' = 0\}, dt''$	F_{d22}

Let $\beta^* H_2$ be the lift of H_2 to $\mathcal{H}^3_c$, and let $\beta^* K_2$ be the lift of K_2 to $\mathcal{H}^3_c$. Then, $\beta^* K_2$ vanishes to infinite order at all boundary faces except those arising from the lift of F_{0101}. Now let

$$H_3 := \beta_*(\beta^* H_2 - \beta^* H_2 \beta^* K_2),$$

where β_* is the push forward to $\mathcal{H}$ from $\mathcal{H}^3_c$. Since $\beta^* K_2$ vanishes to infinite order at all boundary faces except F_{0101}, the push forward of $(\beta^* H_2)(\beta^* K_2)$ to $\mathcal{H}$ vanishes to infinite order at all boundary faces except F_{0101}, where the result is given by the conic heat calculus composition rule. Consequently,

$$H_3 = H_2 - \beta_*(\beta^* H_2 \beta^* K_2)$$

vanishes to higher order at the boundary faces of F_{0101}, by the conic heat calculus composition rule. Continuing this construction and using Borel summation, we arrive at H_∞ with expansion asymptotic to $H_2, H_3, \ldots$ and satisfying

$$(\partial_\tau + \Delta_0) H_\infty = K_\infty,$$

where K_∞ now vanishes to infinite order on F_{0101}. Using a smooth cutoff function we now have H_∞ defined on all of $\mathcal{H}$ satisfying

$$(\partial_t + \Delta_\epsilon) H_\infty = K_\infty$$

where K_∞ vanishes to infinite order at all boundary faces of $\mathcal{H}$.

7.3.7 Solving away the residual error term

To complete this construction we must remove the residual error term which vanishes to infinite order at the boundary faces of $\mathcal{H}$. It is now convenient to consider the elements of the acc heat calculus as t-convolution operators acting on $\mathcal{S} \times \mathbb{R}^+$. For an element A which vanishes to infinite order at the boundary faces of $\mathcal{H}$ and u a smooth half density section of $\mathcal{S} \times \mathbb{R}^+$, the t-action of A on u is

$$Au(t) = \int_0^t \langle Au(t-s), u(s) \rangle ds, \tag{9}$$

where the spatial variables have been suppressed. As a function of $s', s \geq 0$, $[Au(s')](s)$ vanishes to infinite order at $s' = 0$. Restricting to $s' = t - s$,

$$[Au(t-s)](s) = s^{-k/2-1}(t-s)^j u_{k,j}(t-s, s),$$

for any $-k, j \in \mathbb{N}_0$ with $u_{k,j}$ a smooth half density, so for any $-k \geq 1$ this is integrable and consequently, $Au(t)$ as in (9) is smooth in t and vanishes rapidly as $t \to 0$. So, an element A of the acc heat calculus which vanishes to infinite order at all boundary faces of $\mathcal{H}$ gives rise to a Volterra operator. Since as a t-convolution operator we have

$$(\partial_t + \Delta) H_\infty = Id - K_\infty,$$

we would like to invert $(Id - K_\infty)$. Formally, the inverse should be

$$(Id - K_\infty)^{-1} = \sum_{j \geq 0} K_\infty^j,$$

where K_∞^j is the j-fold composition of K_∞. To show that this Neumann series converges, we estimate the kernel of K_∞^j. Since K_∞ vanishes to infinite order at all boundary faces of $\mathcal{H}$ we may restrict to submanifolds of $\mathcal{H}$, estimating as in [26] and then combine these estimates to estimate K_∞^j on $\mathcal{H}$. The kernel k_∞^j of the restriction of K_∞^j to $M \times M \times \mathbb{R}^+ \times \{\epsilon\}$ is bounded by

$$|k_\infty^j(z, z', t, \epsilon)| \leq C_{\epsilon,j} \frac{t^j}{(j+1)!}, \quad t < T.$$

This follows from the composition rule for the heat calculus on M and the analogous bound in [26] 7.3, where we have taken $k = -2$, with k as above which we are free to choose since A vanishes to infinite order. Similarly, by the composition rule for the heat calculus on M_0 and the same estimate of [26], the kernel of the restriction of K_∞^j to $M_0 \times M_0 \times \mathbb{R}^+$ is bounded by

$$|k_\infty^j(z, z', t)|_{F0101} \leq C_{0,j} \frac{t^j}{(j+1)!}, \quad t < T.$$

Similarly, the kernel of the restriction of K_∞^j to $\bar{Z} \times \bar{Z} \times \mathbb{R}^+$ is bounded by

$$|k_\infty^j(z, z', t)|_{F1010} \leq C_{z,j} \frac{t^j}{(j+1)!}, \quad t < T.$$

These three bounds imply that the constants $C_{\epsilon,j}$ stay bounded as $\epsilon \to 0$ and so we have the following global bound for the kernel of K_∞^j on both $\mathcal{H}$ and the blown-down space $\{\epsilon = \epsilon'\} \subset \mathcal{S} \times \mathcal{S} \times \mathbb{R}^+$

$$|k_\infty^j(z, z', t, \epsilon)| \leq C_j \frac{t^j}{(j+1)!}, \quad t < T.$$

It follows that the Neumann series for $(Id - K_\infty)^{-1}$ is summable and has an inverse which as a t-convolution operator is also of the form $(Id - A)$ where A is an element of the acc heat calculus that vanishes to infinite order at the boundary faces of $\mathcal{H}$. Then, the full acc heat kernel is

$$H = H_\infty (Id - K_\infty)^{-1}.$$

As a consequence of this construction, H has a fully polyhomogeneous expansion down to $\epsilon = 0$ with leading order terms given by the acc model heat kernel, H_1.

♡

Remarks

A consequence of this theorem is the convergence for $t > 0$, $t < T$

$$H_\epsilon \to H_0 + O(\epsilon) \text{as } \epsilon \to 0.$$

This extends the convergence results of [8] for scalar heat kernels to heat kernels for geometric Laplacians acting on vector bundles. Due to the rescaled time variable at the faces contained in $\epsilon, t = 0$, we expect to see something interesting when we compute the short time asymptotic behavior of the heat trace.

A Asymptotically conic scattering heat kernel

Let $\bar{Z}$ be a compactified ac scattering space with boundary defined by $\{x = 0\}$ and local coordinates (x, y) near the boundary. Let Δ_z be the Friedrich's extension of a geometric Laplacian on Z. We motivate the definition of the acc heat space by lifting the Euclidean heat kernel to $\bar{Z}^2_+$.

Recall the Euclidean heat kernel for $\mathbb{R}^n$,

$$G(z, z', t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|z-z'|^2}{2t}}.$$

Here the coordinate $z = (r, y)$ has not been compactified. With the compactification of Z given by $x = \frac{1}{r}$ in the local coordinates (x, y, x', y', t) on $\bar{Z}^2_+$ near the boundary of $\bar{Z}$ the Euclidean heat kernel is

$$G(x, y, x', y', t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{\left(|\frac{1}{x} - \frac{1}{x'}|^2 + |y - y'|^2\right)}{2t}\right).$$

This motivates blowing up

$$S_{110} = \{(x, y, x', y', t) \,:\, x = 0, x' = 0\}.$$

In the projective coordinates $s = \frac{x}{x'}$, $s' = x'$ the Euclidean heat kernel is

$$G(s, y, s', y', t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{\left(|\frac{s-1}{ss'}|^2 + |y - y'|^2\right)}{2t}\right).$$

This motivates a second blowup at $s = 1$, along the submanifold where the diagonal in $\bar{Z} \times \bar{Z}$ meets the first blown up face

$$S_{220} = \{(x, y, x', y', t) \,:\, x = 0, x' = 0, y = y'\}.$$

A.1 The Ac scattering heat space

As motivated above, the ac scattering heat space is constructed from $\bar{Z}^2_+$ by performing three blowups.

First, the scattering double space, $\bar{Z}^2_{sc}$ is constructed:

$$\bar{Z}^2_{sc} := \left[[\bar{Z} \times \bar{Z}; \partial\bar{Z} \times \partial\bar{Z}]; \Delta(Y \times Y) \cap F_{110}\right]$$

where F_{110} is the face created by the first blowup. Including the time variable we perform one more blowup to construct the ac scattering heat space,

$$\bar{Z}^2_{sc,h} = \left[\bar{Z}^2_{sc} \times \mathbb{R}^+; \Delta(Z \times Z) \times \{t = 0\}, dt\right].$$

The ac scattering heat space has six boundary faces described in the following table.

Face	Geometry of face	Defining function in local coordinates
F_{220}	$N^+(\Delta(Y \times Y)) \times \mathbb{R}^+$	$\rho_{220} = (x^2 + (x')^2 + \lvert y - y'\rvert^2)^{\frac{1}{2}}$
F_{220}	$N^+((Y \times Y) - \Delta(Y \times Y)) \times \mathbb{R}^+$	$\rho_{110} = (x^2 + (x')^2)^{\frac{1}{2}}$
F_{100}	$Z \times Y \times \mathbb{R}^+$	$\rho_{100} = x$
F_{010}	$Y \times Z \times \mathbb{R}^+$	$\rho_{010} = x'$
F_{d2}	$PN_t^+(\Delta(Z \times Z))$	$\rho_{d2} = (\lvert z - z'\rvert^4 + t^2)^{\frac{1}{2}}$
F_{001}	$(Z \times Z) - \Delta(Z \times Z)$	$\rho_{001} = t$

A.2 Ac scattering heat calculus

Elements of the ac scattering heat calculus are distributional section half densities on Z_+^2 which are smooth on the interior and lift to be polyhomogeneous on $\bar{Z}^2_{sc,h}$. Let μ be a smooth, non-vanishing half density on $\bar{Z} \times \bar{Z} \times \mathbb{R}^+$ and let ν be a smooth, non-vanishing half density on $\bar{Z}^2_{sc,h}$.

Definition 14. *For any $k \in \mathbb{R}$ and index sets E_{110}, E_{220}, A in$\Psi_{sc,H}^{E_{110},E_{220},k}$ if the following hold.*

1. *$A \in \mathcal{A}_{phg}^{-\frac{1}{2}+E_{110}}$ at F_{110}.*
2. *$A \in \mathcal{A}_{phg}^{-\frac{n+2}{2}+E_{220}}$ at F_{220}.*
3. *A vanishes to infinite order at F_{001}, F_{100}, and F_{010}.*
4. *$A \in \rho_{d2}^{-\frac{n+3}{2}-k}\mathcal{C}^\infty(F_{d2})$.*

Elements of the ac scattering heat calculus are Schwartz kernels of operators acting on sections of Z in the usual way and on sections of $Z \times \mathbb{R}_t^+$ by t-convolution. The composition rule is proven using the ac scattering triple heat space, $\bar{Z}^3_{sc,h}$. This space has partial blow down/projection maps to three identical copies of the ac scattering heat space as well as full blow down/projection maps to three identical copies of $\bar{Z}^2_+$; these are called the left, right, and center. Formally, two elements of the ac heat calculus are composed by lifting from the left and right copies of $\bar{Z}^2_{sc,h}$ to $\bar{Z}^3_{sc,h}$, multiplying and blowing down/projecting to the center copy of $\bar{Z}^2_{sc,h}$. It is key that the triple space be constructed so that these lifts and push-forward maps are b-fibrations in order that polyhomogeneity be preserved.

A.2.1 The ac scattering triple heat space

We first construct the ac scattering triple space $\bar{Z}^3_{sc}$ and later include the time variables. In a neighborhood of the boundary in each copy of $\bar{Z}$ we have the local coordinates (x, y), which provide the local coordinates (x, y, x', y', x'', y'') on $\bar{Z}^3$. First we blow up the codimension three corner defined by $\{x = 0, x' = 0, x'' = 0\}$. We call this face F_{11100} with defining function locally given by

$$\rho_{11100} = (x^2 + (x')^2 + (x'')^2)^{\frac{1}{2}}.$$

Next, we blow up the three codimension two corners corresponding to the F_{110} faces in each of the three copies of $\bar{Z}^2_{sc,h}$. These faces are as follows.

Face	Submanifold to be blown up	Defining Function
F_{11000}	$S_{11000} = \{x = 0, x' = 0\} - F_{11100}$	$\rho_{11000} = (x^2 + (x')^2)^{\frac{1}{2}}$
F_{01100}	$S_{01100} = \{x' = 0, x'' = 0\} - F_{11100}$	$\rho_{01100} = ((x')^2 + (x'')^2)^{\frac{1}{2}}$
F_{10100}	$S_{10100} = \{x = 0, x'' = 0\} - F_{11100}$	$\rho_{10100} = ((x)^2 + (x'')^2)^{\frac{1}{2}}$

Next we blow up the codimension $2n+1$ corner where the diagonals meet F_{11100}. After the F_{11100} blowup, we have coordinates $(\theta, \theta', \theta'', y, y', y'', \rho_{11100})$, with

$$x = (\rho_{11100})\theta, \quad x' = (\rho_{11100})\theta', \quad x'' = (\rho_{11100})\theta'', \quad (\theta)^2 + (\theta')^2 + (\theta'')^2 = 1.$$

Using these coordinates, we next blow up

$$S_{22200} = \{\theta = \theta' = \theta'',\, y = y' = y'',\, r_0 = 0\}.$$

The face created by this blowup is called F_{22200} with defining function

$$\rho_{22200} = ((\theta - \theta')^2 + (\theta' - \theta'')^2 + |y - y'|^2 + |y' - y''|^2 + r_0^2)^{\frac{1}{2}}.$$

After this we blow up the three codimension n corners corresponding to the F_{220} faces in the three copies of the double heat space. These are as follows.

Face	Submanifold to be blown up	Defining Function
F_{22000}	$S_{22000} = \{\theta = 0, \theta' = 0, y = y'\}$	$\rho_{22000} = (\theta^2 + (\theta')^2 + \lvert y - y'\rvert^2)^{\frac{1}{2}}$
F_{02200}	$S_{02200} = \{\theta' = 0, \theta'' = 0, y' = y''\}$	$\rho_{02200} = ((\theta')^2 + (\theta'')^2 + \lvert y' - y''\rvert^2)^{\frac{1}{2}}$
F_{20200}	$S_{20200} = \{\theta = 0, \theta'' = 0, y = y''\}$	$\rho_{20200} = ((\theta)^2 + (\theta'')^2 + \lvert y - y''\rvert^2)^{\frac{1}{2}}$

We have now constructed the ac scattering triple space, $\bar{Z}^3_{sc}$. We next introduce the time variables and perform the parabolic temporal diagonal blowups. We must first blow up the codimension 2 corner of $\mathbb{R}^+ \times \mathbb{R}^+$ to preserve symmetry. Let

$$\mathcal{T}_0^2 = [\mathbb{R}^+ \times \mathbb{R}^+;\, t = t' = 0].$$

The defining function for the blowup of $\{t = t' = 0\}$ is ρ_{00011}, which we call t'' because it plays the role of the third time variable. We now take $Z^3_{sc} \times \mathcal{T}_0^2$ and blow up the temporal diagonal faces. First, we blow up the codimension $2n+3$ triple diagonal, S_{d3}, defined by

$$\{z = z' = z'',\, t'' = 0\}.$$

The defining function of this face is ρ_{d3},

$$\rho_{d3} = (|z - z'|^4 + |z - z''|^4 + (t'')^2)^{\frac{1}{4}}.$$

Next, we blow up the three temporal diagonals corresponding to the diagonal faces in the three copies of the double heat space. These are as follows.

Face	Submanifold to be blown up	Defining Function
F_{d20}	$S_{d20} = \{z = z'\}$	$\rho_{d20} = (\lvert z - z'\rvert^4 + t^2)^{\frac{1}{4}}$
F_{d02}	$S_{d02} = \{z' = z''\}$	$\rho_{d02} = (\lvert z' - z''\rvert^4 + (t')^2)^{\frac{1}{4}}$
F_{d22}	$S_{d22} = \{z = z''\}$	$\rho_{d22} = (\lvert z - z''\rvert^4 + (t'')^2)^{\frac{1}{4}}$

We have now constructed the ac scattering triple heat space and proceed with the composition rule.

Technical Theorem 5. *Let* $A \in \Psi_{sc,H}^{A_{110},A_{220},k_a}$*, and* $B \in \Psi_{sc,H}^{B_{110},B_{220},k_b}$.

Then, the composition $B \circ A$ *is an element of* $\Psi_{sc,H}^{A_{110}+B_{110},A_{220}+B_{220},k_a+k_b}$.

A.2.2 Proof

Formally we have,

$$\kappa_{B\circ A}\nu = (\beta_C)_* \left((\beta_R)^*(\kappa_A \nu)(\beta_L)^*(\kappa_B \nu)\right). \tag{10}$$

Multiplying both sides of (10) by ν and using the fact that $(\beta_c)_*(\beta_c)^*(\nu) = \nu$

$$\kappa_{B\circ A}\nu^2 = (\beta_C)_* \left((\beta_R)^*(\kappa_A \nu)(\beta_L)^*(\kappa_B \nu)(\beta_c)^*(\nu)\right). \tag{11}$$

Next we calculate the lifts of the defining functions and half densities from $\bar{Z}^2_{sc,h}$ to $\bar{Z}^3_{sc,h}$. A calculation gives the half density on the heat space ν in terms of the half density μ on $\bar{Z}^2_+$

$$\nu = (\beta_h)^* \left((\rho_{110})^{-\frac{1}{2}}(\rho_{220})^{-\frac{n}{2}}(\rho_{d2})^{-\frac{n+1}{2}}\mu\right).$$

The ac scattering triple heat space has partial blow down/projection maps β_L, β_R, and β_C to three identical copies of $\bar{Z}^2_{sc,h}$. If we denote the three copies of $\bar{Z}$ by $\bar{Z}$, $\bar{Z}'$, $\bar{Z}''$, and the three time variables (t, t', t'') where t'' is from the blowup of $\mathbb{R}^+ \times \mathbb{R}^+$ then the three copies of $\bar{Z}^2_{sc,h}$ are as follows.

Copy of $\bar{Z}^2_{sc,h}$	Associated to in $\bar{Z}^3_{sc,h}$
Left	$\bar{Z} \times \bar{Z}' \times \mathbb{R}^+_t$
Right	$\bar{Z}' \times \bar{Z}'' \times \mathbb{R}^+_{t'}$
Center	$\bar{Z} \times \bar{Z}'' \times \mathbb{R}^+_{t''}$

Next, we compute the lifts of the defining functions for the boundary faces of the heat space to the triple heat space.

Lifting map	Defining function on $\bar{Z}^2_{sc,h}$	Lift to $\bar{Z}^3_{sc,h}$
$(\beta_L)^*$	ρ_{100}	$\rho_{10000}\rho_{10100}$
$(\beta_L)^*$	ρ_{010}	$\rho_{01000}\rho_{01100}$
$(\beta_L)^*$	ρ_{110}	$\rho_{11100}\rho_{11000}$
$(\beta_L)^*$	ρ_{220}	$\rho_{22200}\rho_{22000}$
$(\beta_L)^*$	ρ_{d2}	$\rho_{d3}\rho_{d20}$
$(\beta_L)^*$	ρ_{001}	$\rho_{00010}\rho_{00011}\rho_{d22}$
$(\beta_R)^*$	ρ_{100}	$\rho_{01000}\rho_{01100}$
$(\beta_R)^*$	ρ_{010}	$\rho_{00100}\rho_{10100}$
$(\beta_R)^*$	ρ_{110}	$\rho_{11100}\rho_{01100}$
$(\beta_R)^*$	ρ_{220}	$\rho_{22200}\rho_{02200}$
$(\beta_R)^*$	ρ_{d2}	$\rho_{d3}\rho_{d02}$
$(\beta_R)^*$	ρ_{001}	$\rho_{00001}\rho_{00011}\rho_{d22}$
$(\beta_C)^*$	ρ_{100}	$\rho_{10000}\rho_{11000}$
$(\beta_C)^*$	ρ_{010}	$\rho_{001000}\rho_{01100}$
$(\beta_C)^*$	ρ_{110}	$\rho_{11100}\rho_{10100}$
$(\beta_C)^*$	ρ_{220}	$\rho_{22200}\rho_{20200}$
$(\beta_C)^*$	ρ_{d2}	$\rho_{d3}\rho_{d22}$
$(\beta_C)^*$	ρ_{001}	$\rho_{00022}\rho_{00011}\rho_{d22}$

Then,

$$(\beta_L)^*(\nu) = (\beta_L)^*((\rho_{110})^{-\frac{1}{2}}(\rho_{220})^{-\frac{n}{2}}(\rho_{d2})^{-\frac{n+1}{2}}\mu).$$

Next, we use the fact that

$$(\beta_L)^*(\mu)(\beta_R)^*(\mu)(\beta_C)^*(\mu) = \mu_3^2.$$

Here, μ_3^2 is a smooth density on $\bar{Z} \times \bar{Z} \times \bar{Z} \times \mathbb{R}^+ \times \mathbb{R}^+$, so we may assume

$$\mu_3^2 = \mathrm{d}z\mathrm{d}z'\mathrm{d}z''\mathrm{d}t\mathrm{d}t'.$$

A Jacobian calculation gives the lift of μ_3^2 to the triple heat space. First note

$$(\beta_3)^*(x) = (\rho_{11100})(\rho_{11000})(\rho_{10100})(\rho_{10000}),$$

$$(\beta_3)^*(x') = (\rho_{11100})(\rho_{11000})(\rho_{01100})(\rho_{01000}),$$

$$(\beta_3)^*(x'') = (\rho_{11100})(\rho_{01100})(\rho_{10100})(\rho_{00100}).$$

This implies

$$(\beta_3)^*(\mu_3^2) = (\rho_{11100})^2(\rho_{11000}\rho_{01100}\rho_{10100})(\rho_{22000}\rho_{02200}\rho_{20200})^n$$

$$(\rho_{22200})^{2n+1}(\rho_{d20}\rho_{d02}\rho_{d22})^{n+1}\rho_{d3}^{2n+3}(t'')\nu_3^2.$$

Here, ν_3^2 is a smooth, nonvanishing density on the triple heat space. Combining this with the above lifts, we arrive at the following formula

$$(\beta_L)^*(\nu)(\beta_R)^*(\nu)(\beta_C)^*(\nu) = (\rho_{11100})^{\frac{1}{2}}(\rho_{10100}\rho_{01100}\rho_{10100})^{\frac{1}{2}}$$
$$(\rho_{22000}\rho_{02200}\rho_{20200})^{\frac{n}{2}}(\rho_{22200})^{\frac{n+1}{2}}(\rho_{d3})^{\frac{n+3}{2}}(\rho_{d20}\rho_{d02}\rho_{d22})^{\frac{n+1}{2}}(t'')\nu_3^2.$$

To use the push forward theorem of [23], we need to write each of these in terms of b-densities. First, we have on the center copy of $\bar{Z}^2_{sc,h}$

$${}^b\nu^2 = (\rho_{100}\rho_{010}\rho_{110}\rho_{220}\rho_{001}\rho_{d2})^{-1}\nu^2.$$

Then, we have

$${}^b\nu^2 = (\beta_c)_*(\beta_c)^*((\rho_{100}\rho_{010}\rho_{110}\rho_{220}\rho_{001}\rho_{d2})^{-1}\nu^2).$$

We observe

$$(\beta_c)^*\left((\rho_{100}\rho_{010}\rho_{110}\rho_{220}\rho_{001}\rho_{d2})^{-1}\right) =$$
$$(\rho_{10000}\rho_{00100}\rho_{11000}\rho_{01100}\rho_{10100}\rho_{11100}\rho_{22200}\rho_{02200}\rho_{20200}\rho_{d3}\rho_{d22}\rho_{00011})^{-1}.$$

So now we multiply both sides of (11) by $(\beta_c)_*(\beta_c)^*(\rho_{100}\rho_{010}\rho_{110}\rho_{220}\rho_{001}\rho_{d2})^{-1})$ and inside the right side of (11) we have

$$(\rho_{11100}\rho_{11000}\rho 01100\rho_{10100})^{-\frac{1}{2}}(\rho_{22000}\rho_{02200}\rho_{22200})^{\frac{n}{2}}(\rho_{20200})^{\frac{n-2}{2}}$$

$$(\rho_{d3})^{\frac{n+1}{2}}(\rho_{d20}\rho_{d02})^{\frac{n+1}{2}}(\rho_{d22})^{\frac{n}{2}}(\rho_{10000}\rho_{00100})^{-1}\nu_3^2.$$

To use the push forward theorem, we must change the density ν_3^2 to a b-density. We observe

$${}^b\nu_3^2 = (\rho_{11100}\rho_{11000}\rho_{01100}\rho_{10100}\rho_{22200}\rho_{22000}\rho_{02200}\rho_{20200}$$
$$\rho_{10000}\rho_{01000}\rho_{00100}\rho_{d3}\rho_{d20}\rho_{d02}\rho_{d22}\rho_{00011}\rho_{00010}\rho_{00001})^{-1}\nu_3^2.$$

So, we now have for the composition formula

$$(\beta_c)_*(\tilde{\kappa_A}\tilde{\kappa_B}(\rho_{11100}\rho_{11000}\rho_{01100}\rho_{10100})^{\frac{1}{2}}(\rho_{22200}\rho_{22000}\rho_{02200})^{\frac{n+2}{2}}$$
$$(\rho_{20200})^{\frac{n}{2}}(\rho_{d3}\rho_{d20}\rho_{d02})^{\frac{n+3}{2}}(\rho_{d22})^{\frac{n+1}{2}}\rho_{01000}\rho_{00011}\rho_{00010}\rho_{00001}({}^b\nu_3^2)).$$

We observe the following orders of $\tilde{\kappa_A}$ on $\bar{Z}^3_{sc,h}$.

Face	$\tilde{\kappa_A}$ Index Set/Leading Order
F_{11100}	$-\frac{1}{2} + A_{110}$
F_{11000}	$-\frac{1}{2} + A_{220}$
F_{01100}, F_{10100}, F_{02200}, F_{20200}, F_{d22}	∞
F_{22200}, F_{22000}	$-\frac{n+2}{2} + A_{220}$
F_{d3}, F_{d20}	$-\frac{n+3}{2} - k_a$
F_{10000}, F_{01000}, F_{00100}, F_{00010}, F_{00011}	∞

Similarly, for $\tilde{\kappa_B}$ we have orders as follows.

Face	$\tilde{\kappa_B}$ Index Set/Leading Order
F_{11100}	$-\frac{1}{2}+B_{110}$
F_{01100}	$-\frac{1}{2}+B_{220}$
F_{11000}, F_{10100}, F_{22000}, F_{20200}, F_{d22}	∞
F_{22200}, F_{02200}	$-\frac{n+2}{2}+B_{220}$
F_{d3}, F_{d02}	$-\frac{n+3}{2}-k_b$
F_{10000}, F_{01000}, F_{00100}, F_{00010}, F_{00011}	∞

Now, recalling the formula:

$$(\beta_c)_*(\tilde{\kappa_A}\tilde{\kappa_B}(\rho_{11100}\rho_{11000}\rho_{01100}\rho_{10100})^{\frac{1}{2}}(\rho_{22200}\rho_{22000}\rho_{02200})^{\frac{n+2}{2}}$$

$$(\rho_{20200})^{\frac{n}{2}}(\rho_{d3}\rho_{d20}\rho_{d02})^{\frac{n+3}{2}}(\rho_{d22})^{\frac{n+1}{2}}\rho_{01000}\rho_{00011}\rho_{00010}\rho_{00001}({}^b\nu_3^2))$$

We see that the quantity on the right hand side to be pushed forward by $(\beta_c)_*$ has the following indices on the boundary faces.

Face	Index Set/Leading Order
F_{11100}	$-\frac{1}{2}+A_{110}+B_{110}$
F_{11000}, F_{01100}, F_{10100}, F_{22000}, F_{02200}, F_{20200}	∞
F_{22200}	$-\frac{n+2}{2}+A_{220}+B_{220}$
F_{d3}	$-\frac{n+3}{2}-(k_a+k_b)$
F_{d20}, F_{d02}, F_{d22}	∞
F_{10000}, F_{01000}, F_{00100}, F_{00010}, F_{00001}, F_{00011}	∞

The push forward under $(\beta_c)^*$ sends the boundary faces of $\bar{Z}^3_{sc,h}$ to $\bar{Z}^2_{sc,h}$ as follows.

$\bar{Z}^3_h$ Face	Boundary face of $\bar{Z}^2_{sc,h}$ or Interior
F_{11100}	F_{110}
F_{10100}	F_{110}
F_{22200}, F_{20200}	F_{220}
F_{d3}, F_{d22}	F_{d2}
F_{10000}	F_{100}
F_{00100}	F_{010}
F_{00011}	F_{001}
F_{11000}, F_{01100} F_{22000}, F_{02200},	Interior
F_{d20}, F_{d02}, F_{01000}, F_{00010}, F_{00001}	Interior

The quantity to be pushed forward is integrable with respect to ${}^b\nu_3^2$ at the faces that are mapped to the interior, so we may apply the push forward theorem (see [23]) to arrive at the result of the composition rule. The kernel, $\kappa_{B\circ A}$ will have the following polyhomogeneous index sets and leading orders on $\bar{Z}^2_h$.

Face of $\bar{Z}^2_h$	Index Set/Leading Order
F_{110}	$-\frac{1}{2} + A_{110} + B_{110}$
F_{220}	$-\frac{n+2}{2} + A_{220} + B_{220}$
F_{d2}	$-\frac{n+3}{2} - (k_a + k_b)$
F_{100}	∞
F_{010}	∞
F_{001}	∞

This concludes the proof of the composition rule: $B \circ A$ is an element of $\Psi_{ac,H}^{A_{110}+B_{110},A_{220}+B_{220},k_a+k_b}$.

♡

Technical Theorem 6. *Let (Z, g_z) be an asymptotically conic scattering space with cross section (Y, h) at infinity. Let (E, ∇) be a Hermitian vector bundle over (Z, g_z) so that near the boundary E is the pullback of a bundle over (Y, h). Let Δ be a geometric Laplacian on (Z, g_z) associated to the bundle (E, ∇). Then there exists $H \in \Psi_{ac,H}^{E_{110},E_{220},-2}$ satisfying:*

$$(\partial_t + \Delta)H(z, z', t) = 0,\ t > 0,$$
$$H(z, z', 0) = \delta(z - z').$$

Moreover, H vanishes to infinite order at F_{110} and is smooth up to F_{220}.

On the interior of $\bar{Z}^2_{ac,h}$ the ac scattering model heat kernel is locally defined by the Euclidean heat kernel and a partition of unity. At F_{d2} we construct the model heat kernel explicitly using the jet of the metric at the base point of each fiber. At F_{001} the model heat kernel vanishes to infinite order. At F_{110} and F_{220} the model heat kernel is the lift of the Euclidean heat kernel. Then the ac scattering model heat kernel H_1 satisfies

$$(\partial_t + \Delta)H_1 = K_1,$$

where K_1 vanishes to positive order on the boundary faces of $\bar{Z}^2_{sc,h}$. We now define

$$H_2 = H_1 - H_1 K_1,$$

with

$$(\partial_t + \Delta)H_2 = K_2$$

where K_2 vanishes to one order higher on the boundary faces of $\bar{Z}^2_{sc,h}$. Similarly,

$$H_3 := H_2 - H_2 K_2.$$

Using Borel summation we construct H_∞ with expansion asymptotic to $H_1, H_2, H_3, \ldots$ and satisfying

$$(\partial_t + \Delta)H_\infty = K,$$

where now K vanishes to infinite order on the boundary faces of $\bar{Z}^2_{sc,h}$. As a t-convolution operator we wish to have

$$H_\infty = Id,$$

however, we currently have

$$H_\infty = Id + K,$$

but this is not a problem since $(Id + K)$ is invertible with inverse of the same form. Then the ac scattering heat kernel

$$H = H_\infty (Id - K)^{-1}$$

is an element of the ac scattering heat calculus with leading orders on the boundary faces of $\bar{Z}^2_{sc,h}$ determined by those of the model kernel.

♡

B Acc triple heat space

Let

$$T := [[[[\mathcal{S} \times \mathcal{S} \times \mathcal{S}; Y \times Y' \times Y'']; Y \times Y']; Y' \times Y'']; Y \times Y''],$$

where we have used Y, Y', Y'' to denote the three copies of Y in $\mathcal{S}^3$. Let

$$\mathcal{T} := \{p \in T : f_i(p) = 0,\ i = 1, 2\},\ f_1(p) = x(p)r(p) - x'(p)r'(p),\ f_2(p) = x'(p)r'(p) - x''(p)r''(p).$$

Like the acc double and heat space, $\mathcal{T}$ is a smooth manifold with corners. Let

$$\mathbb{R}^+_{2,b} := [\mathbb{R}^+_t \times \mathbb{R}^+_s; \{0\} \times \{0\}].$$

Then the acc triple heat space is constructed from $\mathcal{T} \times \mathbb{R}^+_{2,b}$ by blowing up along twelve submanifolds creating the following twelve boundary faces. Below, let tD be the lift to $\mathcal{T}$ of the diagonal in $\mathcal{S} \times \mathcal{S}' \times \mathcal{S}''$, let D_{110} be the lift of the diagonal in $\mathcal{S} \times \mathcal{S}'$, D_{011} be the lift of the diagonal in $\mathcal{S}' \times \mathcal{S}''$ and D_{101} be the lift of the diagonal in $\mathcal{S} \times \mathcal{S}''$.

Submanifold blown up	Face created
$Y \times Y' \times Y'' \times \{0\} \times \{0\}, dt, ds$	S_{11122}
$Y \times Y' \times \{t = 0\}, dt$	S_{11020},
$Y' \times Y'' \times \{s = 0\}, ds$	S_{01102}
$Y \times Y'' \times \{t = s = 0\}, ds, dt$	S_{10122}
$Y \times Y' \times Y''$	S_{111}
$Y \times Y'$	S_{110}
$Y' \times Y''$	S_{011}
$Y \times Y''$	S_{101}
$tD \times \{t, s = 0\}, ds, dt$	S_{td}
$D_{110} \times \{t = 0\}, dt$	S_{d20}
$D_{011} \times \{s = 0\}, ds$	S_{d02}
$D_{101} \times \{s = t = 0\}, ds, dt$	S_{d22}

As constructed, the acc triple heat space has full and partial projection/blow down maps to three identical copies, left, right and center, of the acc heat space and to three corresponding copies of the blown down space $\{\epsilon = \epsilon'\} \subset \mathcal{S}^2 \times \mathbb{R}^+$. To compose two elements A and B we view the element A as acting from the left to the right while B acts from the right to the center. Formally, the composition $B \circ A$ is the pushforward from the acc triple heat space of the product of the lifts of A and B. Compatibility assumptions on the leading orders of A and B at boundary faces of the acc heat space are required so that we can push forward. With these assumptions and with the possible inclusion of normalizing factors at boundary faces of the acc heat space, two elements compose as one would expect. The technical details in the proof of this composition rule are expected to be analogous to the technical details in the proof of the ac scattering heat calculus composition rule (appendix A).

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