A Generalization of Boolean Rings

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Abstract: A Boolean ring satisfies the identity $x^2 = x$ which, of course, implies the identity $x^2y - xy^2 = 0$. With this as motivation, we define a *subBoolean* ring to be a ring R which satisfies the condition that $x^2y - xy^2$ is nilpotent for certain elements x, y of R. We consider some conditions which imply that the subBoolean ring R is commutative or has a nil commutator ideal.

Throughout, R is a ring, not necessarily with identity, N the set of nilpotents, C the center, and J the Jacobson radical of R. As usual, $[x, y]$ will denote the commutator $xy - yx$.

Definition. A ring R is called *subBoolean* if

$$(1) \qquad x^2y - xy^2 \in N \text{ for all } x,\ y \text{ in } R \setminus (N \cup J \cup C).$$

The class of subBoolean rings is quite large, and contains all Boolean rings, all commutative rings, all nil rings, and all rings in which $J = R$. On the other hand, a subBoolean ring need not be Boolean or even commutative. Indeed, the ring

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} : 0, 1 \in GF(2) \right\}$$

is subBoolean but not Boolean and not commutative. Theorem 6 below gives a characterization of *commutative* subBoolean rings.

In preparation for the proofs of the main theorems, we need the following two lemmas.

Lemma 1 ([1]) *Suppose R is a ring in which each element x is central, or potent in the sense that $x^k = x$ for some integer $k > 1$. Then R is commutative.*

Lemma 2 *If R is a subBoolean ring with central idempotents, then the set N of nilpotents is contained in the Jacobson radical J of R.*

Proof. Suppose $a \in N$, $x \in R$. Suppose for the momenet that $ax \in (N \cup J \cup C)$. If $ax \in N$, then ax is right quasiregular (r.q.r.). Also, $ax \in J$ implies that ax is r.q.r. Now suppose $ax \in C$ (the center of R). Then $(ax)^m = a^m x^m$ for all positive integers m, and hence $ax \in N$ (since $a \in N$), which again implies that ax is r.q.r. Next, consider the case $(ax)^2 \in (N \cup J \cup C)$. Again, $(ax)^2 \in N$ implies that ax is r.q.r., while $(ax)^2 \in C$ implies $(ax)^{2k} = (ax)^2(ax)^2 \cdots (ax)^2 = a^k t$ for some $t \in R$, which implies that $ax \in N$ (since $a \in N$), and hence ax is r.q.r. Finally, if $(ax)^2 \in J$, then $(ax)^2$ is r.q.r., and hence ax is r.q.r. Combining the above facts, we have:

(2) $$\text{If } ax \in (N \cup J \cup C) \text{ or } (ax)^2 \in (N \cup J \cup C), \text{ then } ax \text{ is r.q.r.}$$

Now, suppose $ax \notin (N \cup J \cup C)$ and $(ax)^2 \notin (N \cup J \cup C)$. Then, by (1),

(3) $$\left((ax)^2\right)^2 (ax) - (ax)^2(ax)^2 \in N.$$

In view of (3), we see that

$$(ax)^q = (ax)^{q+1} g(ax)\,; \quad g(\lambda) \in \mathbb{Z}[\lambda]\,;\ q \geq 1.$$

Let $e = [(ax)g(ax)]^q$. Then $e^2 = e$, and $(ax)^q = (ax)^q e$. Hence,

(4) $$(ax)^q = (ax)^q e;\ e = [(ax)g(ax)]^q\,;\ e^2 = e; \quad (a \in N).$$

Suppose $a^m = 0$ (recall that $a \in N$). Since the idempotents are central, (4) readily implies

$$e = ee = e\,[(ax)g(ax)]^q = eat = aet, \text{ for some } t \text{ in } R,$$

and thus $e = aet = a^2 e t^2 = \cdots = a^m e t^m = 0$. Hence, by (4), $ax \in N$, and thus ax is r.q.r. The net result is:

(5) $$\text{If } ax \notin (N \cup J \cup C) \text{ and } (ax)^2 \notin (N \cup J \cup R), \text{ then } ax \text{ is } \textit{r.q.r.}$$

Combining (2) and (5), we conclude that ax is r.q.r. for all x in R, and hence $a \in J$, which proves the lemma.

We are now in a position to prove our main theorems.

Theorem 1 *If R is a subBoolean ring with central idempotents, then R/J is commutative.*

Proof. By Lemma 2, $N \subseteq J$, and hence by (1),

(6) $$x^2 y - x y^2 = 0 \text{ for all } \textit{noncentral} \text{ elements } x,\ y \text{ in } R/J.$$

Since the semisimple ring R/J is isomorphic to a subdirect sum of primitive rings R_i $(i \in \Gamma)$, each of which satisfies (6), we have

$$x^2y - xy^2 = 0 \text{ for all } \textit{noncentral} \text{ elements } x,\ y \text{ in } R_i,\ (i \in \Gamma). \tag{7}$$

Case 1. R_i is a division ring. Suppose R_i is *not* commutative. Let x_i be a *noncentral* element of R_i. Then, by (7), $x_i^2(x_i + 1) - x_i(x_i + 1)^2 = 0$, and hence $x_i = 0$ or $x_i = -1$, a contradiction which proves that R_i is commutative.

Case 2. R_i is a primitive ring which is not a division ring. In this case, by Jacobson's density theorem [3, p.33], there exists a division ring D and an integer $k > 1$ such that the complete matrix ring D_k satisfies (7). This, however, is false, as can be seen by taking $x = E_{12}$, $y = E_{12} + I_k$; x, y in D_k. This contradiction shows that Case 2 nevers occurs, which forces R_i to be a division ring, and hence R_i is commutative (see Case 1). This proves the theorem.

Theorem 2 *Suppose R is a reduced ($N = \{0\}$) ring and R is a subBoolean ring. Suppose, further, that J is commutative. Then R is commutative.*

Proof. Since R is reduced, all idempotents are central, and hence by Theorem 1, R/J is commutative. Therefore, since J is commutative,

$$[[x, y], [z, t]] = 0 \text{ for all } x, y, z, t \text{ in } R. \tag{8}$$

Note that (8) is a polynomial identity which is satisfied by all elements of R. However, (8) is *not* satisfied by any 2×2 complete matrix ring over $GF(p)$ for any prime p, as can be seen by taking $[x, y] = [E_{11}, E_{12}]$, $[z, w] = [E_{22}, E_{21}]$. Hence, by [2], the commutator ideal of R is nil, and thus R is commutative (since $N = \{0\}$).

Corollary 1 *A Boolean ring is commutative.*

This follows at once from Theorem 2, since the Jacobson radical of a Boolean ring is $\{0\}$.

Corollary 2 *Suppose R is a ring with identity, and suppose R is reduced and subBoolean. Then R is commutative.*

Proof. Let $j, j' \in J$ and suppose $[j, j'] \neq 0$. Then, by (1),

$$(1 + j)^2(1 + j') - (1 + j)(1 + j')^2 \in N = \{0\},$$

and hence $(1 + j)\{(1 + j) - (1 + j')\}(1 + j') = 0$, which implies that (since $1 + j$ and $1 + j'$ are units in R), $j = j'$, contradiction. This contradiction proves that J is commutative, and the corollary follows from Theorem 2.

Theorem 3 *Suppose R is a subBoolean ring with central idempotents, and suppose $J \subseteq C$. Then R is commutative.*

Proof. By Lemma 2, $N \subseteq J$ and hence $N \subseteq J \subseteq C$, which, when combined with (1), yields

$$x^2y - xy^2 \in N \quad \text{for } all\ x,\ y \text{ in } R \setminus C. \tag{9}$$

Suppose $x \notin C$. Setting $y = -x$ in (9) yields $2x^3 \in N$, and hence $2x \in N \subseteq C$ (see above). Thus,

$$2x \in C \text{ for all } x \text{ in } R. \tag{10}$$

Next, we prove that

$$x^2 \in C \text{ for all } x \text{ in } R. \tag{11}$$

To see this, recall that by Theorem 1, $[x, y] \in J \subseteq C$, and hence $[x, y]$ is central for all x, y in R. Using this fact and (10) yields

$$[x^2, y] = x[x, y] + [x, y]x = 2x[x, y] = x[2x, y] = 0,$$

which proves (11). We prove Theorem 3 by contradiction. Suppose $x \notin C$ for some $x \in R$. Then $x + x^2 \notin C$ (see (11)), and hence by (9),

$$x^2(x + x^2) - x(x + x^2)^2 \in N, \text{ and thus } x^3(x + x^2) \in N. \tag{12}$$

Therefore, for some polynomial $g(\lambda) \in \mathbb{Z}[\lambda]$, we have

$$(x + x^2)^4 = (x + x^2)^3(x + x^2) = (x^3g(x))(x + x^2) = x^3(x + x^2)g(x). \tag{13}$$

Note that the right side of (13) is a sum of pairwise commuting nilpotent elements (see (12)), and hence by (13), $x + x^2 \in N \subseteq C$ (see above). Therefore, using (11), we conclude that $x \in C$, contradiction. This proves the theorem.

Theorem 4 *Suppose R is a subBoolean ring with identity and with central idempotents. Suppose, further, that J is commutative. Then R is commutative.*

Proof. By Lemma 2, $N \subseteq J$. We claim that

$$J \subseteq N \cup C. \tag{14}$$

Suppose not. Let $j \in J$, $j \notin N$, $j \notin C$. Since $N \subseteq J$, (1) implies

$$x^2y - xy^2 \in N \text{ for all } x, y \in R \setminus (J \cup C). \tag{15}$$

Note that $1+j \notin J \cup C$, and $J^2 \subseteq C$ (since J is commutative). Therefore, $1+j+j^2 \notin J \cup C$, and hence by (15),

$$(1+j+j^2)^2(1+j) - (1+j+j^2)(1+j)^2 \in N,$$

which implies $j^2(1+j+j^2)(1+j) \in N$. Since $(1+j+j^2)^{-1}$ and $(1+j)^{-1}$ are units in R, and since they both commute with j, it follows that $j^2 \in N$, and hence $j \in N$, contradiction. This contradiction proves (14). In view of (14) and (1), we have

$$x^2y - xy^2 \in N \text{ for all } x,\ y \text{ in } R \setminus (N \cup C). \tag{16}$$

Now, suppose $x \notin N$, $x+1 \notin N$, $x \notin C$ (and hence $x+1 \notin C$). Then, by (16), we see that $x^2(x+1) - x(x+1)^2 \in N$, and thus $x(x+1) \in N$. Since $x \in N$ or $x+1 \in N$ implies that $x(x+1) \in N$, we conclude that

$$x + x^2 = x(x+1) \in N \text{ for all } x \in R \setminus C. \tag{17}$$

Since $x \in C$ implies $-x \in C$, we may repeat the above argument with x replaced by $(-x)$ to get (see (17))

$$x - x^2 \in N \text{ for all } x \in R \setminus C. \tag{17$'$}$$

As is well-known,

$$R \cong \text{a subdirect sum of subdirectly irreducible rings } R_i \ (i \in \Gamma).$$

Let $\sigma : R \to R_i$ be the natural homomorphism of R onto R_i, and let $\sigma : x \to x_i$. We claim that

(18) The set N_i of nilpotents of R_i is contained in $\sigma(N) \cup C_i$,

where C_i denotes the center of R_i. To prove this, let $d_i \in N_i$, $d_i \notin C_i$, and let $\sigma(d) = d_i$, $d \in R$. Then $d \notin C$, and hence by (17)$'$, $d - d^2 \in N$. Since d_i is nilpotent, let $d_i^k = 0$, and observe that (since $d - d^2 \in N$),

$$d - d^{k+1} = (d - d^2)(1 + d + d^2 + \cdots + d^{k-1}) \in N,$$

which implies that $\sigma(d - d^{k+1}) \in \sigma(N)$. Thus $d_i - d_i^{k+1} \in \sigma(N)$, and hence $d_i \in \sigma(N)$, which proves (18). Our next goal is to prove that

(19) Every element of R_i is nilpotent or a unit or central.

To prove this, let $x_i \in R_i \setminus C_i$, and suppose $\sigma(x) = x_i$, $x \in R$. Then $x \notin C$, and hence by $(17)'$, $x - x^2 \in N$, and thus $x^q = x^{q+1}g(x)$ for some $g(\lambda) \in \mathbb{Z}[\lambda]$. The last equation implies that $x^q = x^q[xg(x)]^q$ and $[xg(x)]^q = e$ is idempotent. Therefore

(20) $$x^q = x^q e;\ e = [xg(x)]^q;\ e^2 = e.$$

This reflects in R_i as follows:

(21) $$x_i^q = x_i^q e_i;\ e_i = [x_i g(x_i)]^q;\ e_i^2 = e_i.$$

Since, by hypothesis, the idempotents of R are central, it follows that $e_i = \sigma(e)$ is a *central* idempotent in the subdirectly irreducible ring R_i, and hence $e_i = 1$ or $e_i = 0$. If $e_i = 0$, then by (21), x_i is nilpotent. On the other hand, if $e_i = 1$, then again by (21), x_i is a unit in R, which proves (19). Next, we prove that

(22) Every unit u_i in R_i is central or $u_i = 1 + a_i$ for some nilpotent element a_i in N_i.

To prove this, suppose u_i is a *unit* in R_i, which is *not* central, and suppose $\sigma(d) = u_i$, $d \in R$. Then $d \in R \setminus C$, and hence by $(17)'$, $d - d^2 \in N$, which implies that $u_i - u_i^2 \in \sigma(N)$. Therefore, $u_i^2 - u_i$ is nilpotent; say, $(u_i^2 - u_i)^m = 0$. Hence, $(u_i - 1)^m = 0$, and thus $u_i - 1 = a_i$, a_i nilpotent; that is, $u_i = 1 + a_i$, $a_i \in N_i$, and (22) is proved.

Returning to (18), note that since $N \subseteq J$ (Lemma 2) and J is commutative (by hypothesis), N itself is a commutative set, and hence by (18), the set N_i of nilpotents of R_i is commutative also. Moreover, by (19) and (22), the ring R_i is generated by its nilpotent and central elements, and hence R_i is commutative, which implies that the ground ring R itself is commutative. This completes the proof.

Theorem 5 *A subBoolean ring with identity and with central nilpotents is necessarily commutative.*

Proof. First, we prove that

(23) The set U of units of R is commutative.

Suppose not. Let u, v be units in R such that $[u, v] \neq 0$. Then, by (1), $u^2v - uv^2 \in N$. Also, since $N \subseteq C$, N is an ideal of R, and hence

$$u^{-1}(u^2v - uv^2)v^{-1} \in N,$$

which implies that $u - v \in N \subseteq C$. Thus, $[u, v] = 0$, contradiction. This contradiction proves (23). Let $j, j' \in J$. Then by (23), $[1 + j, 1 + j'] = 0$, and hence $[j, j'] = 0$; that is, J

is commutative. Furthermore, since all nilpotents are central, the idempotents of R are all central. Therefore, by Theorem 4, R is commutative.

In preparation for the proof of our next theorem, recall that an element x of R is called *potent* if $x^k = x$ for some integer $k > 1$. The ring R is called *subweakly periodic* if every element x in $R \setminus (J \cup C)$ can be written as a sum of a nilpotent and a potent element of R.

We are now in a position to state and prove the next theorem, which characterizes all *commutative* subBoolean rings (compare with Theorem 3.1 of [4]).

Theorem 6 *Suppose R is a subBoolean ring. Suppose, further, that the idempotents of R are central and J is commutative. If, in addition, R is subweakly periodic, then R is commutative (and conversely).*

Proof. To begin with, if zero is the *only* potent element of R, then (by definition of a subweakly periodic ring), $R = N \cup J \cup C = J \cup C$ (since $N \subseteq J$, by Lemma 2), and hence R is commutative, since J is commutative. Thus, we may assume that R has a *nonzero* potent element. Let a be any nonzero potent element of R, and let $a^k = a$ with $k > 1$. Let $e = a^{k-1}$. Then e is a nonzero idempotent which, by hypothesis, is *central.* Hence, eR is a ring *with identity.* Moreover, eR is a subBoolean ring (keep in mind that the Jacobson radical of eR is eJ, where J is the Jacobson radical of R). Also, the idempotents of eR are central, and the Jacobson radical of eR (namely, eJ) is commutative. Hence, by Theorem 4, eR is commutative. Let $y \in R$. Then $e[a, y] = [ea, ey] = 0$. Recalling that $e = a^{k-1} \in C$ and $a^k = a$, it follows that

$$0 = e[a, y] = a^{k-1}[a, y] = a^k y - a^{k-1} ya = a^k y - ya^k = ay - ya,$$

for all y in R, and hence

(24) All potent elements of R are central.

To complete the proof, let $x, y \in R \setminus (J \cup C)$ for the moment. Then

(25) $$x = a + b, \quad y = a' + b'; \quad a, a' \in N; \quad b, b' \text{ potent.}$$

By Lemma 2, $N \subseteq J$, and hence by (25),

(25)′ $$x = a+b, \quad y = a'+b'; \quad a, a' \in J; \quad b, b' \text{ potent.}$$

Therefore, by (24) and the hypothesis that J is commutative,

$$[x, y] = [a + b, a' + b'] = [a, a'] = 0 \qquad \text{(see (25′))}.$$

By a similar argument, $[x, y] = 0$ also if $x \in J \cup C$ or $y \in J \cup C$. This completes the proof.

A concept related to commutativity is the notion that the commutator ideal is nil. In this connection, we have the following theorem.

Theorem 7 *Suppose R is a subBoolean ring with identity and with central idempotents. Then the commutator ideal of R is nil.*

Proof. First we prove that

$$J \subseteq N \cup C. \tag{26}$$

Suppose not. Let $j \in J$, $j \notin N$, $j \notin C$. Then $1 + j \notin J$, $1 + j \notin N$, $1 + j \notin C$. We now distinguish two cases.

Case 1. $j^2 \notin C$. In this case, $1 + j^2 \notin J$, $1 + j^2 \notin N$, $1 + j^2 \notin C$. Hence, by (1),

$$(1 + j)^2(1 + j^2) - (1 + j)(1 + j^2)^2 \in N,$$

and thus $j(1 - j^4) \in N$. Since $(1 - j^4)^{-1}$ is a unit in R which commutes with j, it follows that $j \in N$, contradiction.

Case 2. $j^2 \in C$. In this case, a similar argument shows that, since $1+j+j^2 \notin (N \cup J \cup C)$ and $1 + j \notin (N \cup J \cup C)$,

$$(1 + j)^2(1 + j + j^2) - (1 + j)(1 + j + j^2)^2 \in N,$$

which implies $j^2(1 + j)(1 + j + j^2) \in N$. Since $[(1 + j)(1 + j + j^2)]^{-1}$ is a unit in R which commutes with j^2, it follows that $j^2 \in N$, and hence $j \in N$, contradiction. This contradiction (in both cases) proves (26).

Next, we prove that

$$N \text{ is an ideal of } R. \tag{27}$$

By Lemma 2, $N \subseteq J$, which when combined with (26) yields

$$N \subseteq J \subseteq N \cup C. \tag{28}$$

Now, suppose $a \in N$, $b \in N$. Then, by (28), $a \in J$, $b \in J$, and hence $a - b \in J \subseteq (N \cup C)$ (see (28)), which implies $a - b \in N$ or $a - b \in C$, and thus $a - b \in N$ (in either case). Next, suppose $a \in N$, $x \in R$. Then, by (28), $a \in J$, and hence $ax \in J \subseteq (N \cup C)$, which implies $ax \in N$ or $ax \in C$. If $ax \in C$, then $(ax)^k = a^k x^k$ for all $k \geq 1$, and hence $ax \in N$ (since $a \in N$). So in either case, $ax \in N$. Similarly $xa \in N$, which proves (27).

Returning to (26), we see that $N \cup J \cup C = N \cup C$, which when combined with (1) shows that

$$x^2y - xy^2 \in N \text{ for all } x, y \in R \setminus (N \cup C). \tag{29}$$

Keeping (27) in mind, we see that (29) implies

(30) $$x^2y - xy^2 = 0 \text{ for all } \textit{noncentral} \text{ elements } x,\ y \text{ in } R/N.$$

Suppose $x \in R/N$ is *noncentral*. Then $x + 1 \in R/N$ is *noncentral* also, and hence by (30), $x^2(x+1) - x(x+1)^2 = 0$. Therefore, $x(1+x) = 0$, which implies that $x(1+x)(1-x) = 0$; that is, $x^3 = x$ (if x is noncentral). The net result is:

(31) Every element of R/N is central or potent (satisfying $x^3 = x$).

It follows, by Lemma 1, that R/N is commutative, and hence the commutator ideal of R is nil. This completes the proof.

We conclude with the following:

Remark. If in the definition of a subBoolean ring (see (1)), we replace the exponent 2 by n, where n is a fixed positive integer other than 2, then neither Theorem 4 nor Theorem 6 is necessarily true. To see this, let

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} : a, b, c \in GF(4) \right\}.$$

It can be verified that R satisfies the condition

$$x^7y - xy^7 \in N \text{ for all } x,\ y \text{ in } R.$$

Furthermore, R satisfies *all* the hypotheses of *both* Theorems 4 and 6 (except, of course, the exponent 2 is now replaced by 7). But R is *not* commutative.

References

[1] BELL, H.E.: *A near-commutativity property for rings.* Result. Math. 42 (2002), 28–31.

[2] BELL, H.E.: *On some commutativity theorems of Herstein.* Arch. Math. 24 (1973), 34–38.

[3] JACOBSON, N.: *Structure of rings.* Amer. Math. Soc. Colloq. Publications, Vol. 37, American Mathematical Society, Providence, RI, 1964.

[4] YAQUB, A.: *On weakly periodic-like rings and commutativity.* Result. Math 49 (2006), 377–386.