On Generalized Periodic-like Rings

Howard E. Bell[†] and Adil Yaqub

Abstract: Let R be a ring with center Z, Jacobson radical J, and set N of all nilpotent elements. Call R generalized periodic-like if for all $x \in R \setminus (N \cup J \cup Z)$ there exist positive integers m, n of opposite parity for which $x^m - x^n \in N \cap Z$. We identify some basic properties of such rings and prove some results on commutativity.

Let R be a ring; and let $N = N(R)$, $Z = Z(R)$ and $J = J(R)$ denote respectively the set of nilpotent elements, the center, and the Jacobson radical. As usual, we call R periodic if for each $x \in R$, there exist distinct positive integers m, n such that $x^m = x^n$. In [3] we defined R to be generalized periodic (g–p) if for each $x \in R \setminus (N \cup Z)$

(∗) there exist positive integers m, n of opposite parity such that $x^m - x^n \in N \cap Z$.

We now define R to be generalized periodic-like (g–p–l) if (∗) holds for each $x \in R \setminus (N \cup J \cup Z)$. Clearly, the class of g–p–l rings contains all commutative rings, all nil rings, all Jacobson radical rings, all g–p rings, and some (but not all) periodic rings. It is our purpose to exhibit some general properties of g–p–l rings and to study commutativity of such rings.

1 Preliminary results

To simplify our discussion, we denote by $((m, n))$ the ordered pair of integers m, n of opposite parity. The rest of our notation and terminology is standard. For elements $x, y \in R$, the symbol $[x, y]$ denotes the commutator $xy - yx$; for subsets $X, Y \subseteq R$, $[X, Y]$ denotes the set $\{[x, y] \mid x \in X,\ y \in Y\}$; and $C(R)$ denotes the commutator ideal of R. An element $x \in R$ is called regular if it is not a zero divisor; it is called periodic if there exist distinct positive

[†]Supported by the Natural Sciences and Engineering Research Council of Canada, Grant 3961.

integers m, n for which $x^m = x^n$; and it is called potent if there exists an integer $n > 1$ for which $x^n = x$. The set of all potent elements of R is denoted by P or $P(R)$, and the prime radical by $\mathfrak{P}(R)$. Finally, R is called reduced if $N(R) = \{0\}$.

Lemma 1.1 *Let R be an arbitrary g–p–l ring.*

(i) Every epimorphic image of R is a g–p–l ring.

(ii) $N \subseteq J$.

(iii) If $[N, J] = \{0\}$, then N is an ideal.

(iv) $C(R) \subseteq J$.

(v) If e is an idempotent, the additive order of which is not a power of 2, then $e \in Z$.

Proof. (i) is clear, once we recall that if $\sigma : R \to S$ is an epimorphism, then $\sigma(J(R)) \subseteq J(S)$.

(ii) Let $S = R/J(R)$. Then by (i), S is a g–p–l ring; and since $J(S) = \{0\}$, S is a g–p ring. It follows from Theorem 1 of [3] that $N(S)$ is an ideal of S, hence $N(S) \subseteq J(S) = \{0\}$ and therefore $N(R) \subseteq J(R)$.

(iii) Since $N \subseteq J$, N is commutative and hence $(N, +)$ is an additive subgroup. Let $a \in N$ and $x \in R$. Then $ax \in J$, so $[a, ax] = 0$ – i.e. $a^2x = axa$. It follows that $(ax)^2 = a^2x^2$ and that $(ax)^n = a^nx^n$ for all positive integers n. Therefore $ax \in N$.

(iv) As in (ii), $R/J(R)$ is a g–p ring; hence, by Lemma 2 of [3], $C(R/J(R)) = \{0\}$. Therefore $C(R) \subseteq J(R)$.

(v) If $e \notin Z$, then $-e \notin J \cup Z$ and there exists $((m, n))$ such that $(-e)^m - (-e)^n \in N \cap Z$. Since m, n are of opposite parity, we get $2e \in N$, so that $2^ke = 0$ for some k.

Lemma 1.2 *Let R be an arbitrary g–p–l ring, and let $x \in R$. Then either $x \in J \cup Z$, or there exists a positive integer q and an idempotent e such that $x^q = x^qe$.*

Proof. If $x \notin J \cup Z$, there exists $((m, n))$ such that $x^m - x^n \in N \cap Z$. Therefore there exists a positive integer q and $g(t) \in \mathbb{Z}[t]$ such that $x^q = x^{q+1}g(x)$. It is now easy to verify that $e = (xg(x))^q$ is an idempotent with $x^q = x^qe$.

Lemma 1.3 *Let R be a g–p–l ring and σ an epimorphism from R to S. Then $N(S) \subseteq \sigma(J(R)) \cup Z(S)$.*

Proof. Let $s \in N(S)$ with $s^k = 0$ and let $d \in R$ such that $\sigma(d) = s$. If $d \in J(R) \cup Z(R)$, then obviously $s \in \sigma(J(R)) \cup Z(S)$; hence we may suppose that there exists $((m, n))$ with $n > m$ such that $d^m - d^n \in N(R) \cap Z(R)$. It is easy to show that $d - d^h \in N$, where $h = n - m + 1$; thus

$$d - d^{k+1} d^{k(h-2)} = d - d^h + d^{h-1}(d - d^h) + \cdots + (d^{h-1})^{k-1}(d - d^h)$$

is a sum of commuting nilpotent elements, hence is in $N(R)$ and therefore in $J(R)$. Consequently $s - s^{k+1} s^{k(h-2)} \in \sigma(J(R))$; and since $s^{k+1} = 0$, $s \in \sigma(J(R))$.

We finish this section by stating two known results on periodic elements.

Lemma 1.4 *Let R be an arbitrary ring, and let $N^* = \{x \in R \,|\, x^2 = 0\}$.*

(i) [1, Lemma 1] If $x \in R$ is periodic, then $x \in P + N$.

(ii) [2, Theorem 2] If N^* is commutative and N is multiplicatively closed, then $PN \subseteq N$.

2 Commutativity results

Theorem 2.1 *If R is a g–p–l ring with $J \subseteq Z$, then R is commutative.*

Proof. Suppose $x \notin Z$. Then by Lemma 1.1 (ii), we have $((m, n))$ with $n > m$ such that $x^m - x^n \in N \cap Z$. Consequently $x^{n-m+1} - x \in N$; and since $N \subseteq Z$, commutativity of R follows by a well-known theorem of Herstein [4].

Theorem 2.2 *If R is any g–p–l ring with 1, then R is commutative.*

Proof. We show that if R is g–p–l with 1, then $J \subseteq Z$. Suppose that $x \in J \setminus Z$. Then $-1 + x \notin J \cup Z$, so there exists $((m, n))$ such that $(-1 + x)^m - (-1 + x)^n \in N \cap Z$; and we may assume that m is even and n is odd. Since $N \subseteq J$, it follows that $2 \in J$; thus for every integer m, $2m \in J$, and hence $2m + 1$ is invertible.

Now consider $((m_1, n_1))$ such that $(1+x)^{m_1} - (1+x)^{n_1} \in N \cap Z$. Then $(m_1 - n_1)x + x^2 p(x) \in N \cap Z$ for some $p(t) \in \mathbb{Z}[t]$; and since $m_1 - n_1$ is central and invertible, we get $x + x^2 w$ in $N \cap Z$ for some w in R with $[x, w] = 0$. Thus, we have a positive integer q and an element y in R such that $[x, y] = 0$ and $x^q = x^{q+1} y$. It follows that $e = (xy)^q$ is an idempotent such that $x^q = x^q e$; and since J contains no nonzero idempotents, x is in N.

Let α be the smallest positive integer for which $x^k \in Z$ for all $k \geq \alpha$, and note that, since $x \notin Z$, $\alpha \geq 2$. But $1 + x^{\alpha-1} \notin J \cup Z$, so there exists $((m_2, n_2))$ such that $(1 + x^{\alpha-1})^{m_2} - (1 + x^{\alpha-1})^{n_2} \in N \cap Z$; hence $(m_2 - n_2)x^{\alpha-1} \in Z$. But since $m_2 - n_2$ is invertible and central, we conclude that $x^{\alpha-1} \in Z$ – a contradiction.

Theorem 2.3 *If R is a reduced g–p–l ring with $R \neq J$, then R is commutative.*

Proof. If $R = J \cup Z$, then $R = Z$ and we are finished. Otherwise, if $x \in R \setminus (J \cup Z)$, there exists $((m, n))$ such that $x^m - x^n \in N \cap Z = \{0\}$; hence x is periodic, and by Lemma 1.4(i) $x \in P$. Thus, $R = P \cup J \cup Z$; and to complete the proof we need only to show that $P \subseteq Z$.

Let $y \in P$, and let $k > 1$ be such that $y^k = y$. Then $e = y^{k-1}$ is an idempotent for which $y = ye$, and $e \in Z$ since $N = \{0\}$. Now eR is an ideal of R, so that $J(eR) = eR \cap J(R)$; hence eR is a g–p–l ring with 1, which is commutative by Theorem 2.2. Therefore $[ey, ew] = 0$ for all $w \in R$; and since $ey = y$ and $e \in Z$, we conclude that $[y, w] = 0$ for all $w \in R$ – i.e. $y \in Z$.

Theorem 2.4 *If R is a g–p–l ring in which J is commutative and all idempotents are central, then R is commutative.*

Proof. We may express R as a subdirect product of subdirectly irreducible rings, each of which is an epimorphic image of R. Let R_α be such a subdirectly irreducible ring, and let $\sigma : R \to R_\alpha$ be an epimorphism. Let $x_\alpha \in R_\alpha$ and let $x \in R$ such that $\sigma(x) = x_\alpha$. By Lemma 1.2, $x \in J(R) \cup Z(R)$ or there exists an idempotent $e \in R$ and a positive integer q such that $x^q = x^q e$. Thus, either $x_\alpha \in \sigma(J(R)) \cup Z(R_\alpha)$ or $x_\alpha^q = x_\alpha^q e_\alpha$, where $e_\alpha = \sigma(e)$ is a central idempotent of R_α. But R_α is subdirectly irreducible, hence if R_α has a nonzero central idempotent, then R_α has 1 and is commutative by Theorem 2.2.

To complete the proof, we need only consider the case that for each $x_\alpha \in R_\alpha$, $x_\alpha \in \sigma(J(R)) \cup Z(R_\alpha) \cup N(R_\alpha)$. Now by Lemma 1.3, $N(R_\alpha) \subseteq \sigma((J(R)) \cup Z(R_\alpha)$; hence $R_\alpha = \sigma(J(R)) \cup Z(R_\alpha)$, which is clearly commutative. Therefore R is commutative.

Theorem 2.4 has two corollaries, the first of which is immediate when we recall Lemma 1.1(v).

Corollary 2.5 *If R is a 2-torsion-free g–p–l ring with J commutative, then R is commutative.*

Corollary 2.6 *Let R be a g–p–l ring containing a regular central element c. If J is commutative, then R is commutative.*

Proof. It suffices to show that $N \subseteq Z$, since this condition implies that idempotents are central. Consider first the case $c \in J$. Then $cJ \subseteq J^2$, which is central since J is commutative. Since c is regular and central, it is immediate that $J \subseteq Z$, so certainly $N \subseteq Z$.

Now assume that $c \notin J$, and suppose that $a \in N \setminus Z$. Then $c + a \notin J \cup Z$, and there exists $((m, n))$ such that $(c + a)^m - (c + a)^n \in N \cap Z$. It follows that $c^m - c^n$ is a sum of commuting nilpotent elements, hence $c^m - c^n \in N$ and there exists q such that $c^q = c^{q+1}p(c)$ for some $p(t) \in \mathbb{Z}[t]$. As before, we get an idempotent e such that $c^q = c^q e$ and $[c, e] = 0$. Now e cannot be a zero divisor, since that would force c to be a zero divisor; therefore R has a regular idempotent – i.e. R has 1. We have contradicted Theroem 2.2, so $N \subseteq Z$ as claimed.

3 Nil-commutator-ideal theorems

Theorem 3.1 *Let R be a g–p–l ring. If $R \neq J$ and N is an ideal, then $C(R)$ is nil.*

Proof. We may assume $R \neq J \cup Z$, since otherwise R is commutative. Let $\bar{R} = R/N$, and let the element $x + N$ of $\bar{R}$ be denoted by $\bar{x}$. We need to show that $\bar{R}$ is commutative – a conclusion that follows from Theorem 2.3 once we show that $J(\bar{R}) \neq \bar{R}$.

Suppose that $J(\bar{R}) = \bar{R}$, and let $x \in R \setminus (J \cup Z)$. By Lemma 1.2, there exists a positive integer q and an idempotent $e \in R$ such that $x^q = x^q e$; and it follows that $\bar{e}$ is an idempotent of $\bar{R}$ such that $\bar{x}^q = \bar{x}^q \bar{e}$. But $\bar{R} = J(\bar{R})$ contains no nonzero idempotents, so that $\bar{x}^q = 0 = \bar{x}$ and hence $x \in N(R)$. This contradicts the fact that $x \notin J \cup Z$, hence $\bar{R} \neq J(\bar{R})$ as required.

Theorem 3.2 *If R is a g–p–l ring and J is commutative, then $C(R)$ is nil.*

Proof. If $R = J$, then R is commutative. If $R \neq J$, N is an ideal by Lemma 1.1(iii) and $C(R)$ is nil by Theorem 3.1.

In fact, we can improve this result as follows:

Theorem 3.3 *Let R be a g–p–l ring with $R \neq J$. If N is commutative, then $C(R)$ is nil.*

This result follows from Theorem 3.1, once we prove our final theorem.

Theorem 3.4 *Let R be a g–p–l ring with $R \neq J$. If N is commutative, then N is an ideal.*

Proof. Again we may assume that $R \neq J \cup Z$. Since N is commutative, N is an additive subgroup of R. To show that $RN \subseteq N$, it is convenient to work with the ring $\bar{R} = R/\mathfrak{P}(R)$. As in the proof of Theorem 3.1, we have $J(\bar{R}) \neq \bar{R}$; and if $\bar{R} = Z(\bar{R})$, then $C(R) \subseteq \mathfrak{P}(R) \subseteq N$. Therefore, we assume that $\bar{R} \neq J(\bar{R}) \cup Z(\bar{R})$. We note that if $x + N = \bar{x} \in N(\bar{R})$, then $x \in N(R)$; consequently $N(\bar{R})$ is commutative and hence is an additive subgroup of $\bar{R}$.

Now $\bar{R}$ is semiprime and therefore $N(\bar{R}) \cap Z(\bar{R}) = \{0\}$. It follows that if $\bar{x} \in \bar{R} \setminus (J(\bar{R}) \cup Z(\bar{R}))$, there exists $((m,n))$ such that $\bar{x}^m = \bar{x}^n$ – i.e., $\bar{x}$ is periodic. Thus $\bar{x} \in P(\bar{R}) + N(\bar{R})$ by Lemma 1.4(i); and by commutativity of $N(\bar{R})$ and Lemma 1.4(ii) we get $\bar{x}N(\bar{R}) \subseteq N(\bar{R})$. Moreover, if $\bar{y} \in Z(\bar{R})$, $\bar{y}N(\bar{R}) \subseteq N(\bar{R})$. Now let $\bar{y} \in J(\bar{R}) \setminus Z(\bar{R})$, and let $\bar{x} \in \bar{R} \setminus (J(\bar{R}) \cup Z(\bar{R}))$. Then $\bar{x} + \bar{y} \notin J(\bar{R})$, hence it is in $\bar{R} \setminus (J(\bar{R}) \cup Z(\bar{R}))$ or in $Z(\bar{R})$; and in either case $(\bar{x} + \bar{y})N(\bar{R})$ and $\bar{x}N(\bar{R})$ are in $N(\bar{R})$, so that $\bar{y}N(\bar{R}) \subseteq N(\bar{R})$. We have shown that $N(\bar{R})$ is an ideal of $\bar{R}$; therefore if $x \in R$ and $a \in N(R)$, $\bar{x}\bar{a} \in N(\bar{R})$ and hence $xa \in N(R)$. Thus, $N(R)$ is an ideal of R.

Remark. There exist noncommutative g–p–l rings with J commutative. An accessible example is $\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in GF(2) \right\}$.

References

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