Minimal genus problem: New approach

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Abstract

The minimal genus problem of connected sums of 4-manifolds and the minimal slice genus of knots in $\mathbb{C}P^2$ are treated. The approach used is twisting operations on knots in S^3.

We give an upper bound of the smooth slice genus of left-handed torus knots in $\mathbb{C}P^2$ and we study the smooth slice genus of the family of $(2, q)$-torus knots in $\mathbb{C}P^2$ for any $q \geq 3$.

T. Lawson conjectured in [23] that the minimal genus of $(m, n) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ is given by $\binom{m-1}{2} + \binom{n-1}{2}$ -this is the genus realized by the connected sum of algebraic curves in each factor.

T. Lawson also conjectured in [23] that if $X = X_1 \# X_2$ is the connected sum of two symplectic 4-manifolds with $b_2^+ \geq 3$, and if $(a, b) \in H_2(X) = H_2(X_1) \oplus H_2(X_2)$ satisfies $a.a \geq 0$ and $b.b \geq 0$, then the minimal genus for this class is the sum of the minimal genus for the class a and the minimal genus for the class b.

We answer these conjectures by the negative.

1 Introduction

Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. In particular, all knots are oriented. Let X be a closed 4-manifold and K a knot in $\partial(X - intB^4) \cong S^3$, where B^4 is an embedded 4-ball in X. If K bounds a properly embedded 2-disk in $X - intB^4$, then K is called a slice knot in X. We adopt here the terminology of Seifert surface for K, for a properly embedded orientable compact surface $S \subset X - intB^4$ bounding K in $\partial(X - intB^4) \cong S^3$. We denote by $g_s(K)$ the minimal genus over all isotopy classes of smooth Seifert surfaces for K lying in $X - intB^4$.

A (p, q)-*torus knot* $T(p, q)$ ($0 < p < q$ and p and q are coprime) is a knot that wraps around the standard solid torus in the longitudinal direction p times and the meridional direction q times, where the linking number of the meridian and longitude is equal to 1 (see D. Rolfsen [31]).

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I would like to thank heartily D. Auckly, R. E. Gompf and K. C. Millett for interesting discussions, and my Ph.D. thesis advisor Y. Mathieu for introducing me to this subject.

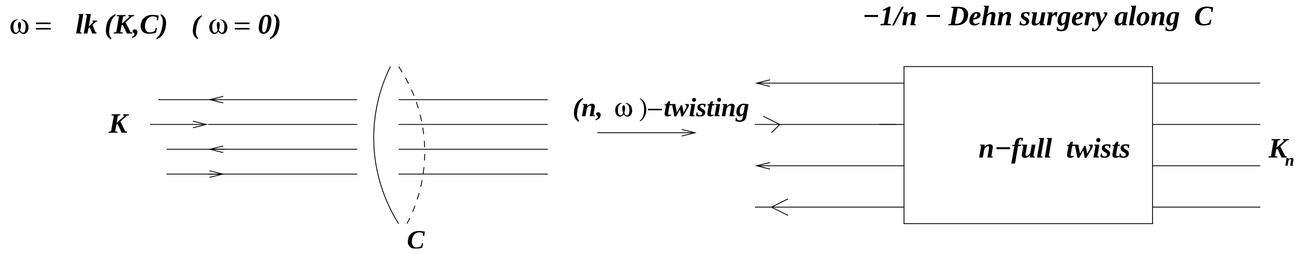

Figure 1:

Let K be a knot in the 3-sphere S^3, and D^2 a disk intersecting K in its interior. Let n be an integer. A $-\frac{1}{n}$-Dehn surgery along ∂D^2 changes K into a new knot K_n in S^3. Let $\omega = \text{lk}(\partial D^2, L)$. We say that K_n is obtained from K by (n, ω)-*twisting* (or simply *twisting*). Then we write $K \overset{(n,\omega)}{\rightarrow} K_n$, or $K \overset{(n,\omega)}{\rightarrow} K(n, \omega)$. We say that K_n is n-twisted provided that K is the unknot (see Figure 1). By Kirby's calculus [19], we can prove that a (-1)-twisted knot in S^3 is smoothly slice in $\mathbb{C}P^2$ (see a proof in [25]). This motivates our interest for studying surfaces in $\mathbb{C}P^2 - intB^4$ bounding torus knots in $\partial(\mathbb{C}P^2 - intB^4) \cong S^3$ in general, and therefore the minimal genus problem in $\mathbb{C}P^2 \# \mathbb{C}P^2$ by the gluing of surfaces techniques.

K. Motegi and K. Miyazaki proved that if a (p, q)-torus knot ($q \neq kp \pm 1$) is n-twisted, then $n = \pm 1$ (see [27]). In addition, if $0 < p < q$ then $n = +1$ (see [4]). Equivalentely, if $T(-p, q)$ ($q \neq kp \pm 1$) is n-twisted, then $n = -1$ and therefore smoothly slice in $\mathbb{C}P^2$ ([2], [25]). Indeed, J. Song and H. Goda and C. Hayashi proved that $T(2, 5)$ and even the family $T(p, p + 2)$ (for $p \geq 9$) are obtained from the unknot by a $(+1)$-twisting (see [13]). This implies that their corresponding left-handed torus knots are smoothly slice in $\mathbb{C}P^2$ (see [2]). We will prove the following:

Proposition 1.1 $T(-p, 4p \pm 1)$ is smoothly slice in $\mathbb{C}P^2$ for any $p \geq 2$.

We will show that $T(-p, 4p \pm 1)$ is $(-1, 2p)$-twisted for any $p \geq 2$ (see Figure 5). R. E. Gompf pointed out, using a different proof, that $T(-2, 7)$ is also smoothly slice in $\mathbb{C}P^2$ ([14]). This can be deduced from Proposition 1.1. We also show that handedness in $\mathbb{C}P^2$ counts e.g. $T(-2, 5)$ is slice in $\mathbb{C}P^2$ but $g_s(T(2, 5)) = 1$ (see Theorem 1.2).

From now on, $g_s(K)$ denotes the minimal genus over all isotopy classes of smooth connected oriented and compact surfaces whose boundary is the knot $K \subset \partial(\mathbb{C}P^2 - intB^4)$, and d denotes its corresponding degree in $H_2(\mathbb{C}P^2 - intB^4, S^3, \mathbb{Z})$. In **section** 2.1, we will prove Theorem 1.1 by explicitely giving a Seifert surface for $T(-p, q)$ lying in $\mathbb{C}P^2 - intB^4$ as stated in Claim 2.1.

Theorem 1.1 $g_s(T(-p, q)) \leq \dfrac{(q-1)(q-p-1)}{2}$.

By an easy application of concordance theory, we can show that a slice knot in S^3 is slice in $\mathbb{C}P^2$. However, the converse is not true since we can easily conclude from Theorem 1.1 that $T(-p, p + 1)$ ($p \geq 2$) is slice in $\mathbb{C}P^2$.

A. Yasuhara [35] proved that there exist an infinite family $T(-2, 2x_i + 1)$ which is non slice in $\mathbb{C}P^2$. However, the value of the smooth slice genus of any non-slice $(\pm p, q)$-torus knot in $\mathbb{C}P^2$ is still unknown.

To answer this question, we will prove in **section** 2.2 the following:

Theorem 1.2 (Handedness)

(1) $g_s(T(-2,5)) = 0$ and $g_s(T(2,5)) = 1$.

(2) $\frac{q \mp 1}{4} \leq g_s(T(2,q)) \leq \frac{q-3}{2}$ for $q \equiv \pm 1 \pmod{4}$.

(3) $g_s(T(-2,7)) = 0$ with $d = 4$, and $g_s(T(2,7)) = 2$ with $d \in \{0, \pm 1\}$.

An interesting question is to find the degree and the smooth slice genus of torus knots in $\mathbb{C}P^2$ in general. Note that $T(p,q)$ is obtained from $T(2,3)$ by adding $(p-1)(q-1)-2$ half-twisted bands. This implies that there is a genus $\frac{(p-1)(q-1)}{2} - 1$ concordance between $T(2,3)$ and $T(p,q)$. We claim that the smooth slice genus in $\mathbb{C}P^2$ and the concordance genus are the same for any (p,q)-torus knot ($0 < p < q$ and p and q are coprime). This let us hit to the following conjecture:

Conjecture 1.1 $g_s(T(p,q)) = \frac{(p-1)(q-1)}{2} - 1$.

All known examples of slice torus knots in $\mathbb{C}P^2$ are (-1)-twisted e.g. $T(-p, 4p \pm 1)$ for any $p \geq 2$ (see Figure 5). Notice that only left-handed torus knots can be slice in $\mathbb{C}P^2$ with the right-handed trefoil as the only exception (see Figure 4). This can be proved by a using a theorem due to P. Gilmer and O. Ya Viro (see Theorem 2.2.1) and a theorem on non-positivity of the signatures of right-handed torus knots in general (see Ait Nouh- Yasuhara [4]). This let us meet with the following conjecture:

Conjecture 1.2 A torus knot is slice in $\mathbb{C}P^2$ if and only if it is (-1)-twisted.

In **section 3**, we disapprove the first Lawson's conjecture by proving the following:

Proposition 3.1 Lawson's conjecture fails for either the pair $(4,1)$ or $(4,-1)$ or $(4,0) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$.

In [5], we answer this conjecture by the positive for the small pairs $(3,3)$ and $(6,6)$.

In **section 4**, we disapprove the second Lawson's conjecture [23] by proving Theorem 1.3.

Let $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ be the 4-manifold equipped with an elliptic fibration, and $E(2) = E(1) \#_f E(1)$ be the fiber sum. We can check that $E(2)$ is a $K3$ surface and then $b_2^+ = 3$ and $b_2^- = 19$ (refer to R. Gompf and A. Stipsicz [15], pp.67 – 76 for more details on elliptic fibrations).

Theorem 1.3 *There exist $(a,b) \in H_2(E(2) \# E(2)) = H_2(E(2)) \oplus H_2(E(2))$ such that $a.a \geq 0$ and $b.b \geq 0$, and the genus of a (resp. b) is minimal in $H_2(E(2))$ (resp. $H_2(E(2))$), but the genus of $a+b$ is less and not equal to the sum of the genus of a and the genus of b.*

The genus function G is defined on $H_2(X, \mathbb{Z})$ as follows: For $\alpha \in H_2(X, \mathbb{Z})$, consider

$$G(\alpha) = min\{genus(\Sigma) | \Sigma \subset X \quad represents \quad \alpha, i.e., [\Sigma] = \alpha\}$$

Where Σ ranges over closed, connected, oriented surfaces smoothly embedded in the 4-manifold X. Note that $G(-\alpha) = G(\alpha)$ and $G(\alpha) \geq 0$ for all $\alpha \in H_2(X, \mathbb{Z})$ (An excellent reference is Gompf-Stipsicz [14]).

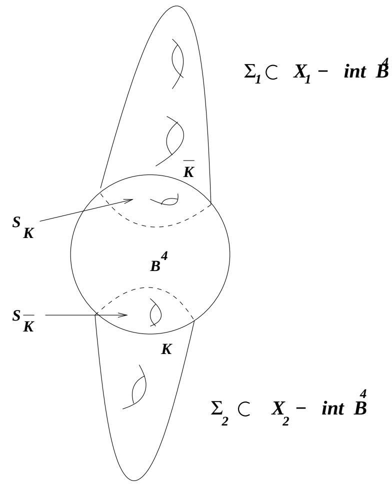

Figure 2: The gluing of surfaces technique

In our setting, the connection between knot theory and dimension four topology is based on the following construction depicted in Figure 2: Let K be a knot in S^3, then the dual knot of K is the inverse of the mirror-image K^* of K i.e. $\overline{K} = -K^*$ ([16]). Denote by X_1^4 and X_2^4 two oriented and closed 4-manifolds and let $(\Sigma_i, \partial\Sigma_i) \subset (X_i - intB^4, S^3)$ for $i = 1, 2$ two compact and oriented surfaces such that $\partial\Sigma_1 = K$ and $\partial\Sigma_2 = \overline{K}$. Denote by $\Sigma_1' = \Sigma_1 \bigcup_K S_K$ and $\Sigma_2' = \Sigma_2 \bigcup_K S_{\overline{K}}$ where S_K (resp. $S_{\overline{K}}$) is the standard Seifert surface for K (resp. $\overline{K}$) in B^4. Gluing Σ_1' and Σ_2' along their boundaries yields a new closed surface $\Sigma_1' \bigcup_K \Sigma_2'$ such that $[\Sigma_1' \bigcup_K \Sigma_2'] = [\Sigma_1'] + [\Sigma_2'] \in H_2(X_1 \# X_2, \mathbb{Z})$ and $g(\Sigma_1' \bigcup_K \Sigma_2') = g(\Sigma_1) + g(\Sigma_2) - g_4(K) - g_4(\overline{K})$, where $g_4(K)$ denotes the 4-ball genus of K.

Let $a = [\Sigma_1'] = [\Sigma_1 \bigcup_K S_K] \in H_2(X_1, \mathbb{Z})$ and $b = [\Sigma_2'] = [\Sigma_1 \bigcup_K S_{\overline{K}}] \in H_2(X_2, \mathbb{Z})$. Then $a + b = [\Sigma_1 \bigcup_K \Sigma_2]$. It is important to notice here that under the assumptions $g_4(K) \geq 1$, and a and b are minimal, then $G(a + b) < G(a) + G(b)$. Indeed, $\Sigma_1 \bigcup_K \Sigma_2$ skips the four ball genus of K and $\overline{K}$. In this fashion, we will present a counterexample to the second Lawson's conjeture as stated in Theorem 1.3 and illustrated in Figure 8 of page 12 with $K = 4_1$ and $X_1 = X_2 = E(2)$ in which we find a and b as described above such that $G(a + b) < G(a) + G(b)$.

If we take the standard connected sum of Σ_1 and Σ_2, then this does not affect the genus. More precisely, we will get a new surface $\Sigma_1 \# \Sigma_2$ whose genus is the sum of the genus of Σ_1 and Σ_2. This proves that if $X = X_1 \# X_2$ is the connected sum of two closed 4-manifolds, and if $(a, b) \in H_2(X) = H_2(X_1) \oplus H_2(X_2)$

then $G(a+b) \leq G(a)+G(b)$. However, the inequality can be strict. Therefore, the minimal genus in a connected sum of 4-manifolds is not always the sum of the minimal genus in each factor.

We mention here that G. Mikhalkin ([28]) has shown that the genus-minimizing surfaces in $\mathbb{C}P^2$ can have their genus reduced further after direct sum with additional copies of $\mathbb{C}P^2$ i.e. $\mathbb{C}P^2\#..\#\mathbb{C}P^2$.

So far, there is no theory for 4-manifolds with even b_2^+, and Seiberg-Witten theory applies mainly to 4-manifolds with odd $b_2^+ > 1$. Connected sums of 4-manifolds with even b_2^+ is an open area of research where gauge theory remains inefficient. In light of the above techniques, we treat $\mathbb{C}P^2\#\mathbb{C}P^2$ and $E(2)\#E(2)$.

2 Proof of statements

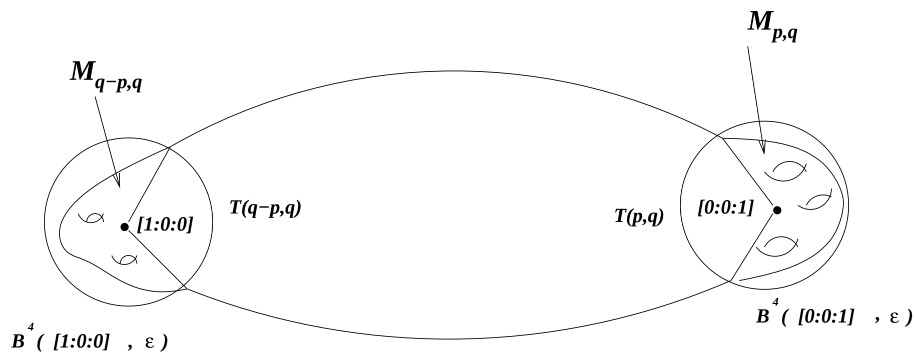

Figure 3: The surfaces Σ and $\tilde{\Sigma}$

2.1 Smooth Seifert surface spanning a $(-p,q)$-torus knot in $\mathbb{C}P^2$

To prove Theorem 1.1, we explicitely give a smooth complex Seifert surface for $T(-p,q)$, and find its genus (see Claim 2.1). Recall some preliminaries: In homogeneous coordinates $[x:y:z]$ where $(x,y,z) \in \mathbb{C}^3$, the complex projective plane $\mathbb{C}P^2$ is covered by three affine charts $U_x := \{[1:y:z] \in \mathbb{C}P^2 | (y,z) \in \mathbb{C}^2\}$, and $U_y := \{[x:1:z] \in \mathbb{C}P^2 | (x,z) \in \mathbb{C}^2\}$ and $U_z := \{[x:y:1] \in \mathbb{C}P^2 | (x,y) \in \mathbb{C}^2\}$. Let Σ be the curve in $\mathbb{C}P^2$ that is given in homogeneous coordinates by $x^p z^{q-p} + y^q = 0$ ($0 < p < q$; p and q are coprime). This curve has two singularities: the one in U_z at $[x:y:z] = [0:0:1]$ whose link is $T(p,q)$, and the other one in U_x at $[1:0:0]$ whose link is $T(q-p,q)$ (see Figure 3). Thus the intersection number with the $\mathbb{C}P^1$ $(y=0)$ is $p+(q-p) = q$ as required. Since Σ has degree q, we can desingularize it by perturbing its equation to obtain a smooth curve $\tilde{\Sigma}$. By Thom's conjecture, that is proved by P. Kronheimer and T. Mrowka (see [20]), the genus of $\tilde{\Sigma}$ is $(q-1)(q-2)/2$.

Claim 2.1 $M^{\infty}_{p,q} = \tilde{\Sigma} \cap (\mathbb{C}P^2 - int(B^4([0:0:1],\epsilon))$ *(see Figure 3) is a smooth complex Seifert surface for* $T(-p,q)$ *in* $\mathbb{C}P^2$ *whose genus is* $\dfrac{(q-1)(q-2)}{2} - \dfrac{(p-1)(q-1)}{2}$.

Proof Desingularizing the singularity $[0:0:1]$ (resp. $[1:0:0]$) replaces the cone on $T(p,q)$ (resp. $T(q-p,q)$) by its Milnor fiber $M_{p,q}$ (resp. $M_{q-p,q}$), which is the obvious Seifert surface for the torus knot $T(p,q)$ whose genus is $(p-1)(q-1)/2$ (resp. $(q-p-1)(q-1)/2$) (see [21],[6]). Thus, if we undo the perturbation to recover Σ, we must subtract such a term for each singularity: the genus of Σ is then $(q-1)(q-2)/2 - (p-1)(q-1)/2 - (q-p-1)(q-1)/2 = 0$. Thus, Σ is a sphere with two locally knotted points. Since $\tilde{\Sigma} = M^{\infty}_{p,q} \bigcup_{T(p,q)} M_{p,q}$, then $\partial M^{\infty}_{p,q} = M_{p,q} \cap S^3([0:0:1],\epsilon)$. Therefore $M^{\infty}_{p,q}$ bounds $T(-p,q)$, and $M^{\infty}_{p,q}$ is smooth, complex and compact. In addition, $g(M^{\infty}_{p,q}) = g(\tilde{\Sigma}) - g(M_{p,q})$, or equivalentely $g(M^{\infty}_{p,q}) = (q-1)(q-2)/2 - (p-1)(q-1)/2$.

Proof of Theorem 1.1 The proof is an immediate corollary of Claim 2.1.

Remark Notice that the degree d of a genus-minimizing Seifert surface for $T(-p,q)$ is different from q in general. Indeed, $T(-p, 4p \pm 1)$ is slice with $d = 2p$ $(q = 4p \pm 1)$ (see Proposition 1.1). Thus the relative Thom conjecture is false in general.

2.2. Proof of Theorem 1.2

We need some preliminaries derived from old gauge theory:

Theorem 2.2.1 (P. Gilmer and O. Ya. Viro [12], [33]) Let X be an oriented, compact 4-manifold with $\partial X = S^3$, and K a knot in ∂X. Suppose K bounds a surface of genus g in X representing $\xi \in H_2(X, \partial X)$.

(1) If ξ is divisible by an odd prime d, then: $\mid \dfrac{d^2-1}{2d^2}\xi^2 - \sigma(X) - \sigma_d(k) \mid \leq dim H_2(X;\mathbb{Z}_d) + 2g$.

(2) If ξ is divisible by 2, then: $\mid \dfrac{\xi^2}{2} - \sigma(X) - \sigma(k) \mid \leq dim H_2(X;\mathbb{Z}_2) + 2g$.

In the following, let b_2^+ (resp. b_2^-) denotes the dimension of the maximal positive (resp. negative) subspace for the intersection form on $H_2(X,\mathbb{Z})$.

Theorem 2.2.2 (K. Kikuchi [18]) Let X be a closed, oriented and smooth 4-manifold such that: (1) $H_1(X)$ has no 2-torsion; and (2) $b_2^{\pm 1} \leq 3$.

If ξ is a characteristic class of $H_2(X,\mathbb{Z})$ represented by an embedded 2-sphere in X, then: $\xi^2 = \sigma(X)$

Theorem 2.2.3 (D. Acosta [1], R. Fintushel [10], A. Yasuhara [35]) Let X be a smooth closed oriented simply connected 4-manifold with $m = min(b_2^+(X), b_2^-(X))$ and $M = max(b_2^+(X), b_2^-(X))$, and assume that $m \geq 2$. Suppose Σ is an embedded surface in X of genus g so that $[\Sigma]$ is characteristic. Then

$$g \geq \begin{cases} \dfrac{\mid \Sigma.\Sigma - \sigma(X) \mid}{8} + 2 - M & if \quad \Sigma.\Sigma \leq \sigma(X) \leq 0 \quad or \quad 0 \leq \sigma(X) \leq \Sigma.\Sigma \\ \dfrac{9(\mid \Sigma.\Sigma - \sigma(X) \mid)}{8} + 2 - M & if \quad \sigma(X) \leq \Sigma.\Sigma \leq 0 \quad or \quad 0 \leq \Sigma.\Sigma \leq \sigma(X) \\ \dfrac{\mid \Sigma.\Sigma - \sigma(X) \mid}{8} + 2 - m & if \quad \sigma(X) \leq 0 \leq \Sigma.\Sigma \quad or \quad \Sigma.\Sigma \leq 0 \leq \sigma(X) \end{cases}$$

To prove Theorem 1.2., we need the following:

Corollary 2.1 $T(2,5)$ *is not slice in* $\mathbb{C}P^2$

Proof Assume for a contradiction that $T(2,5)$ is slice in $\mathbb{C}P^2$, then there exist a properly embedded disk $\Delta \subset \mathbb{C}P^2 - intB^4 = M_1$ such that $\partial\Delta = T(2,5)$. Let $[\Delta] = d\gamma$, where γ is the standard generator of $H_2(\mathbb{C}P^2 - intB^4, S^3, \mathbb{Z})$. If d is even, then by Theorem 2.2.1, $\mid \dfrac{d^2}{2} - \sigma(T(2,5)) - 1 \mid \leq 1$. By A.G. Tristram [32], $\sigma(T(2,5)) = -4$, and then d satisfies $d^2 + 3 \leq 1$, a contradiction.

Assume now that d is odd. We can check that $T(-2,5)$ is obtained from the unknot $T(-2,1)$ by a single $(-2,2)$-twisting. In [25] and [9], the authors proved using Kirby's calculus on the Hopf link [19], that there exist $D \subset M_1 \# M_2 = S^2 \times S^2 - intB^4 = M_2$ such that $[D] = 2\alpha + 2\beta$ and $\partial D = T(-2,5)$. The sphere $[\Delta \cup D] = d\gamma + 2\alpha + 2\beta \in \mathbb{C}P^2 \# S^2 \times S^2$ is a characteristic class. By Kikuchi's Theorem, $[S^2].[S^2] = \sigma(M^4)$ and then $d^2 + 8 = 1$, a contradiction.

Proof of Theorem 1.2

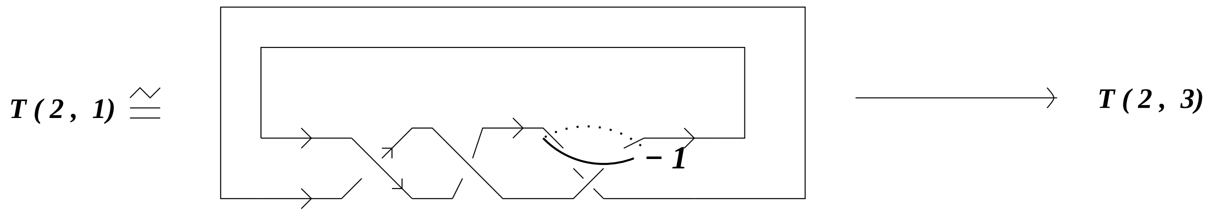

Figure 4:

Notice first that $T(2,3)$ is obtained from the unknot by $(-1,0)$-twisting (see Figure 4), which implies that $T(2,3)$ is smoothly slice in $\mathbb{C}P^2$.

(1) J. Song and H. Goda and C. Hayashi proved in [13] that $T(2,5)$ is obtained from the unknot by a single $(+1,3)$-twisting. Therefore, $T(-2,5)$ is obtained from the unknot by a single $(-1,3)$-twisting ([2]). From [9] and [25] we deduce that $T(-2,5)$ is slice in $\mathbb{C}P^2$. Notice that $T(2,5)$ is obtained from $T(2,3)$ by adding two bands. Thus there is a genus-one cobordism between $T(2,3)$ and $T(2,5)$, and therefore $g_s(T(2,5)) \leq 1$. Corollary 2.1 yields that $g_s(T(2,5)) = 1$.

(2) Assume that $q = 4n \pm 1$ for some integer $n \geq 1$, and prove that $\dfrac{q \mp 1}{4} \leq g_s(T(2,q)) \leq \dfrac{q-3}{2}$.

Case 1 $q = 4n+1$ for some integer $n \geq 1$:

Let $\Sigma_g \subset \mathbb{C}P^2 - intB^4$ be a genus-minimizing g surface such that $\partial\Sigma_g = T(2,4n+1)$ with $[\Sigma_g] = d\gamma$ where γ is the standard generator of $H_2(\mathbb{C}P^2, \mathbb{Z})$. Note that $T(-2,4n+1)$ is obtained from $T(-2,1)$ by a single $(-2n,2)$-twisting. By [9] and [25], there exist a disk $(D, \partial D) \subset (S^2 \times S^2 - intB^4, S^3)$ such that $\partial D = T(-2,4n+1)$ and $[D] = 2\alpha + 2n\beta \in H_2(S^2 \times S^2 - intB^4, S^3)$. The surface $\Sigma = \Sigma_g \cup D \subset \mathbb{C}P^2 \# S^2 \times S^2$ satisfies $[\Sigma] = d\gamma + 2\alpha + 2n\beta \in H_2(\mathbb{C}P^2 \# S^2 \times S^2)$. Thus $[\Sigma]^2 = d^2 + 8n$, so blowing up $\Sigma \subset \mathbb{C}P^2 \# S^2 \times S^2$ a number of times equal to $d^2 + 8n$ gives a genus g surface $\tilde{\Sigma} \subset \mathbb{C}P^2 \# S^2 \times S^2 \# (d^2+8n)\overline{\mathbb{C}P^2}$ (the proper transform) with $[\tilde{\Sigma}]^2 = 0$. If e_i denotes the homology class of the exceptional sphere in the i^{th} blow-up $(i = 1, 2, ..., d^2+8n)$, then $[\tilde{\Sigma}] = d\gamma + 2\alpha + 4\beta - \sum_{i=1}^{i=d^2+8n} e_i$.

If d is odd then $X = \mathbb{C}P^2 \# S^2 \times S^2 \# (d^2+8n)\overline{\mathbb{C}P^2}$ has a signature $\sigma(X) = 1 - d^2 - 8n$. The last inequality of Theorem 2.2.3, yields that $g \geq \dfrac{8n + d^2 - 1}{8}$ $(\star)$, which implies that $g \geq n$.

If d is even, then Gilmer-Viro's Theorem 2.2.1 implies that $\mid \frac{d^2}{2} - 1 - \sigma(T(2,4n+1) \mid \leq 1 + 2g$. Since $\sigma(T(2,4n+1)) = -4n$ (see Tristram [32]), then $\mid \frac{d^2}{2} - 1 + 4n \mid \leq 1 + 2g$, which implies that $2n - 1 \leq g$, and therefore $n \leq g$. Therefore if $q = 4n+1$, then $\frac{q-1}{4} \leq g$.

It is not hard to prove that $g \leq \frac{q-3}{2}$ by induction. Indeed, $T(2,3)$ is slice in $\mathbb{C}P^2$ and there is a genus-two cobordism between $T(2,q)$ and $T(2,q+2)$ and therefore, there is a genus $\frac{q-3}{2}$ between $T(2,3)$ and $T(2,q)$.

Case 2 If $q = 4n-1$ then the proof is similar to Case 1, and we get $\frac{q+1}{4} \leq g \leq \frac{q-3}{2}$

(3) We can deduce from Proposition 1.1, whose proof follows, that $T(-2,7)$ is slice in $\mathbb{C}P^2$ with $d = 4$. Since $T(2,7)$ is abtained from $T(2,3)$, which is slice in $\mathbb{C}P^2$, by adding four half-twisted bands, then $g(T(2,7) \leq 2$. Assume first that d is odd, then letting $n = 2$ in the inequality $(\star)$ yields that $g_s(T(2,7)) = 2$ and $d = \pm 1$. If d is even, then Gilmer-Viro's Theorem 2.2.1 implies that if $g_s(T(2,7)) = 2$ then $d = 0$. Therefore $d \in \{0, \pm 1\}$.

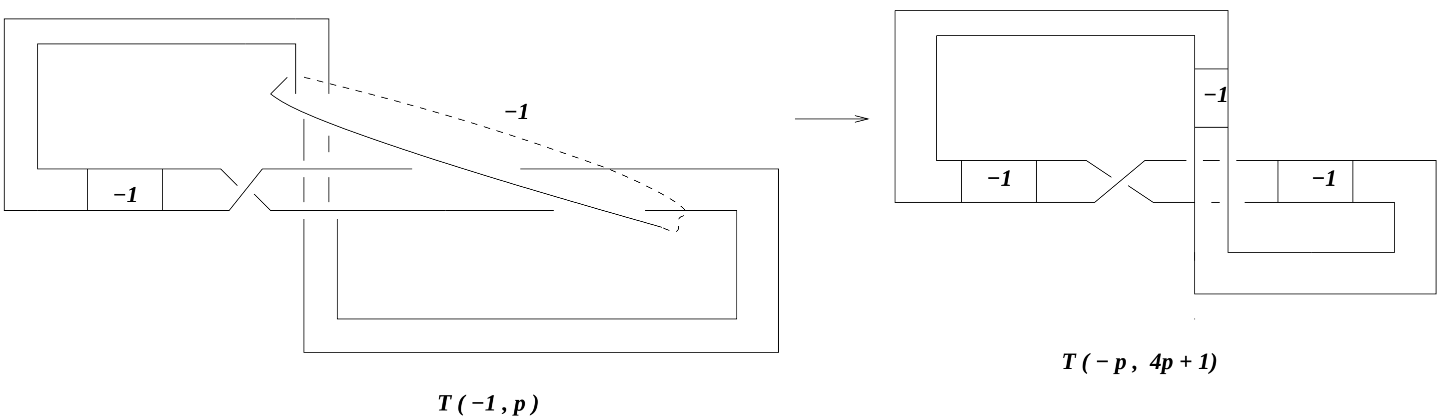

Figure 5:

Proof of Proposition 1.1

Proposition 1.1 $T(-p, 4p \pm 1)$ for $p \geq 2$ is slice in $\mathbb{C}P^2$.

Proof The movie described in Figure 5 proves that $T(-p, 4p+1)$ is obtained from $T(-1,p)$ by a single $(-1, 2p)$-twisting. The proof is similar for $T(-p, 4p-1)$ provided that we start from $T(1,p)$.

3 Minimal genus problem in $\mathbb{C}P^2\#\mathbb{C}P^2$

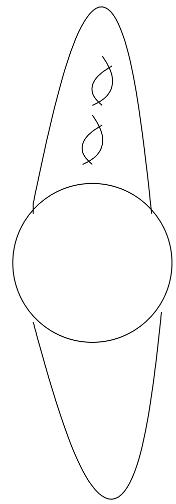

Figure 6:

T. Lawson conjectured in [23] that the minimal genus of $(m,n) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2)$ is given by $\binom{|m|-1}{2} + \binom{|n|-1}{2}$ -this is the genus realized by the connected sum of algebraic curves in each factor. In [5], we answer this conjecture by the positive for the small pairs $(3,3)$ and $(6,6)$. The proofs use twisting of knots in S^3 and gauge theory. We answer here this conjecture by the negative in general.

Proposition 3.1 Lawson's conjecture fails for either the pair $(4,1)$ or $(4,-1)$ or $(4,0) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2)$.

Proof By Proposition 1.1, we deduce that $T(-2,7)$ is slice in $\mathbb{C}P^2$ with degree $d=4$. Therefore, there exist a smooth disk $(\Delta, \partial\Delta) \subset (\mathbb{C}P^2 - intB^4, S^3)$ such that $\partial\Delta = T(-2,7)$ and $[\Delta] = 4\gamma$, where γ is the standard generator of $H_2(\mathbb{C}P^2 - intB^4, S^3)$. By Theorem 1.2, the smooth slice genus of $T(2,7)$ in $\mathbb{C}P^2$ is two. Thus, there exist a genus-two surface $(\Sigma_2, \partial\Sigma_2) \subset (\mathbb{C}P^2 - intB^4, S^3)$ such that $\partial\Sigma_2 = T(2,7)$ and $[\Sigma_2] = d\gamma \in H_2(\mathbb{C}P^2, \mathbb{Z})$ where $d \in \{0, \pm 1\}$. By Theorem 1.2, the genus-two smooth surface $\Sigma = \Delta \cup \Sigma_2$ in $\mathbb{C}P^2\#\mathbb{C}P^2$ satisfies $[\Sigma] = 4\gamma_1 + d\gamma_2 \subset H_2(\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})$ with $d \in \{0, \pm 1\}$ (see Figure 6). If Lawson's conjecture were true, then the genus of Σ which is two should be greater or equal to the proposed Lawson's minimal genus for the pair $(4,d) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})$ which is $3 + \frac{(|d|-1)(|d|-2)}{2}$ where $d \in \{0, \pm 1\}$, a contradiction.

4 Minimal genus problem of connected sum of symplectic surfaces

Let $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ be the 4-manifold equipped with an elliptic fibration, and let F be a regular fiber of $E(1)$. Then a tubular neighborhood of F is $\nu(F) \cong D^2 \times T^2$, and therefore $\partial\nu(F) = T^3 (= \partial(E(1) - \nu(F))$. Define $E(2) = (E(1)-\nu(F))\bigcup_{T^3}(E(1)-\nu(F))$, or simply $E(2) = E(1)\#_F E(1)$ which is called the fiber sum. $E(2)$ is a $K3$ surface and then $b_2^+ = 3$ and $b_2^- = 19$. We have $H_2(E(2), \mathbb{Z}) \cong \mathbb{Z}^{22}$, and a basis is given by 16 spheres $\{S_1, ..., S_{16}\}$ of square -2, realizing $-2E_8$, and three $K3-nucli$ $N_i(2) = N(\sigma_i \cup T_i)(i = 1, 2, 3)$ which can be endowed with a symplectic structure, and such that the intersection matrix of $(\sigma_i, T_i)(i = 1, 2, 3)$ is $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ ([15], p. 72).

Claim 4.1 The intersection matrix of $(\sigma, T, \sigma + 3T)$ is $\begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

Proof By resolving the singular points, $\sigma + 3T$ is a genus three surface. Since $\sigma^2 = -2$, $\sigma T = 1$ and $T^2 = 0$ then $(\sigma + 3T)^2 = \sigma^2 + 6\sigma T + T^2 = 4$, and $\sigma(\sigma + 3T) = \sigma^2 + 3\sigma T = 1$.

Proof of Theorem 1.3

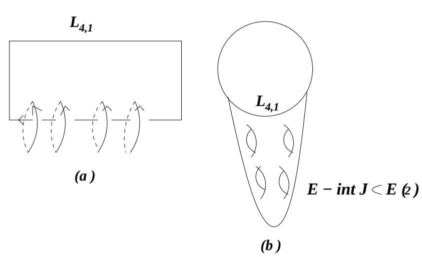

Figure 7:

Claim 4.2 Represent $\sigma + 3T$ by three disjoinct copies of the fiber denoted respectively by T_2, T_3 and T_4. For convenience, we denote $T = T_1$. There exist a surface $E \subset E(2) - int(B^4)$ such that:

- $\partial(E - J) = E \cap \partial J = L_{4,1}$ where the $(4, 1)$-torus link $L_{4,1}$ is depicted in Figure 7(a), and
- $[E - J] = [\sigma] + [T_1] + [T_2] + [T_3] + [T_4]$ in $H_2(E(2) - int(B^4), S^3, \mathbb{Z})$.

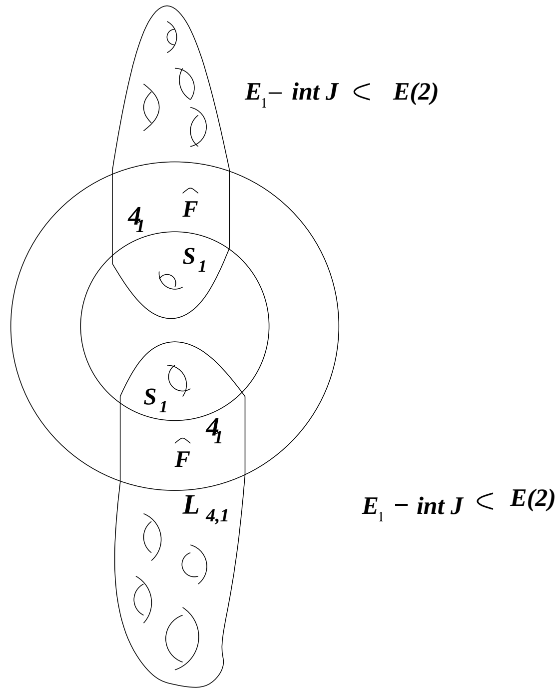

Figure 8:

Proof Consider $E = (\frac{2}{7}, \frac{2}{7}) \times \sigma \cup \bigcup T_1 \times (\frac{3}{7}, \frac{3}{7}) \cup T_2 \times (\frac{4}{7}, \frac{4}{7}) \cup T_3 \times (\frac{5}{7}, \frac{5}{7}) \cup T_4 \times (\frac{6}{7}, \frac{6}{7})$, and the 4-ball $J = [\frac{1}{7}, \frac{6}{7}]^2 \times [\frac{1}{7}, \frac{6}{7}]^2$.

Proof of Theorem 1.3

Notice that the figure eight 4_1 knot is both amphicheiral and invertible, and then $4_1 \cong \overline{4_1}$, where $\overline{4_1}$ is the dual knot of 4_1. By Claim 4.2, there exist a surface E and a 4-ball J, such that: $\partial(E - J) = L_{4,1}$ (see Figure 7(b)). Since 4_1 is obtained from $L_{4,1}$ by fusion (see Figure 7(c)), then there exist a 6-punctured sphere $\hat{F}$ in $S^3 \times [0,1] \subset J$ such that we can identify this band surgery with $\hat{F} \cap (S^3 \times \{1/2\})$, and $\partial\hat{F} = L_{4,1} \cup 4_1$ with $L_{4,1}$ lies in $S^3 \times \{0\} \cong \partial J \times \{0\}$ and 4_1 lies in $S^3 \times \{1\} \cong \partial J \times \{1\}$. By Schőnflies theorem [31], $S^3 \times \{1\} (\cong \partial J \times \{1\})$ bounds a 4-ball $B^4 \subset J$. Let $(S_1, \partial S_1) \subset (intB^4, \partial B^4)$ be a genus one Seifert surface for 4_1 ($g_4(4_1) = 1$), then $\Sigma_1 = (E - int(J)) \bigcup \hat{F} \bigcup S_1$ is represented by $a = [\sigma] + [T] + [\sigma + 3T]$. Since the genus of $E - int(J)$ is four, then the genus of Σ_1 is five. Since the K_3-nucleus is symplectic, then by the adjunction formula $1 + \frac{[\Sigma_1].[\Sigma_1]}{2} = 1 + \frac{8}{2} (= 5)$ (Ozsváth-Szabo [29]). This implies that $a = [\Sigma_1] \in H_2(E(2), \mathbb{Z})$ is genus-minimizing in its homology class. Let Σ_2 be another copy of Σ_1 in $E(2)$, and denote $[\Sigma_2] = b \quad (= a)$. Notice that $a.a = b.b = 8$, and that $[\Sigma_1 \bigcup_{4_1} \Sigma_2] = a + b \in H_2(E(2)\#E(2), \mathbb{Z})$. Therefore the genus of the class $a + b = 2([\sigma] + [T] + [\sigma + 3T])$ is 8 (which is the genus of $\Sigma_1 \bigcup_{4_1} \Sigma_2$). If the second Lawson's conjecture were true, then the homology class of $a + b$ would have genus $5 + 5 = 10$; a contradiction.

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