

# Minimal genus problem: New approach

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## Abstract

The minimal genus problem of connected sums of 4-manifolds and the minimal slice genus of knots in  $\mathbb{C}P^2$  are treated. The approach used is twisting operations on knots in  $S^3$ .

We give an upper bound of the smooth slice genus of left-handed torus knots in  $\mathbb{C}P^2$  and we study the smooth slice genus of the family of  $(2, q)$ -torus knots in  $\mathbb{C}P^2$  for any  $q \geq 3$ .

T. Lawson conjectured in [23] that the minimal genus of  $(m, n) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$  is given by  $\binom{m-1}{2} + \binom{n-1}{2}$  -this is the genus realized by the connected sum of algebraic curves in each factor.

T. Lawson also conjectured in [23] that if  $X = X_1 \# X_2$  is the connected sum of two symplectic 4-manifolds with  $b_2^+ \geq 3$ , and if  $(a, b) \in H_2(X) = H_2(X_1) \oplus H_2(X_2)$  satisfies  $a.a \geq 0$  and  $b.b \geq 0$ , then the minimal genus for this class is the sum of the minimal genus for the class  $a$  and the minimal genus for the class  $b$ .

We answer these conjectures by the negative.

## 1 Introduction

Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. In particular, all knots are oriented. Let  $X$  be a closed 4-manifold and  $K$  a knot in  $\partial(X - \text{int}B^4) \cong S^3$ , where  $B^4$  is an embedded 4-ball in  $X$ . If  $K$  bounds a properly embedded 2-disk in  $X - \text{int}B^4$ , then  $K$  is called a slice knot in  $X$ . We adopt here the terminology of Seifert surface for  $K$ , for a properly embedded orientable compact surface  $S \subset X - \text{int}B^4$  bounding  $K$  in  $\partial(X - \text{int}B^4) \cong S^3$ . We denote by  $g_s(K)$  the minimal genus over all isotopy classes of smooth Seifert surfaces for  $K$  lying in  $X - \text{int}B^4$ .

A  $(p, q)$ -torus knot  $T(p, q)$  ( $0 < p < q$  and  $p$  and  $q$  are coprime) is a knot that wraps around the standard solid torus in the longitudinal direction  $p$  times and the meridional direction  $q$  times, where the linking number of the meridian and longitude is equal to 1 (see D. Rolfsen [31]).

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$$\omega = \text{lk}(K, C) \quad (\omega = 0)$$

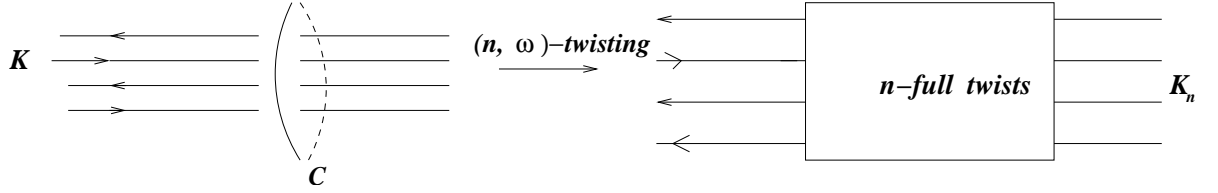


Figure 1:

Let  $K$  be a knot in the 3-sphere  $S^3$ , and  $D^2$  a disk intersecting  $K$  in its interior. Let  $n$  be an integer. A  $-\frac{1}{n}$ -Dehn surgery along  $\partial D^2$  changes  $K$  into a new knot  $K_n$  in  $S^3$ . Let  $\omega = \text{lk}(\partial D^2, L)$ . We say that  $K_n$  is obtained from  $K$  by  $(n, \omega)$ -twisting (or simply *twisting*). Then we write  $K \xrightarrow{(n, \omega)} K_n$ , or  $K \xrightarrow{(n, \omega)} K(n, \omega)$ . We say that  $K_n$  is  $n$ -twisted provided that  $K$  is the unknot (see Figure 1). By Kirby's calculus [19], we can prove that a  $(-1)$ -twisted knot in  $S^3$  is smoothly slice in  $\mathbb{C}P^2$  (see a proof in [25]). This motivates our interest for studying surfaces in  $\mathbb{C}P^2 - \text{int}B^4$  bounding torus knots in  $\partial(\mathbb{C}P^2 - \text{int}B^4) \cong S^3$  in general, and therefore the minimal genus problem in  $\mathbb{C}P^2 \# \mathbb{C}P^2$  by the gluing of surfaces techniques.

K. Motegi and K. Miyazaki proved that if a  $(p, q)$ -torus knot ( $q \neq kp \pm 1$ ) is  $n$ -twisted, then  $n = \pm 1$  (see [27]). In addition, if  $0 < p < q$  then  $n = +1$  (see [4]). Equivalently, if  $T(-p, q)$  ( $q \neq kp \pm 1$ ) is  $n$ -twisted, then  $n = -1$  and therefore smoothly slice in  $\mathbb{C}P^2$  ([2], [25]). Indeed, J. Song and H. Goda and C. Hayashi proved that  $T(2, 5)$  and even the family  $T(p, p+2)$  (for  $p \geq 9$ ) are obtained from the unknot by a  $(+1)$ -twisting (see [13]). This implies that their corresponding left-handed torus knots are smoothly slice in  $\mathbb{C}P^2$  (see [2]). We will prove the following:

**Proposition 1.1**  $T(-p, 4p \pm 1)$  is smoothly slice in  $\mathbb{C}P^2$  for any  $p \geq 2$ .

We will show that  $T(-p, 4p \pm 1)$  is  $(-1, 2p)$ -twisted for any  $p \geq 2$  (see Figure 5). R. E. Gompf pointed out, using a different proof, that  $T(-2, 7)$  is also smoothly slice in  $\mathbb{C}P^2$  ([14]). This can be deduced from Proposition 1.1. We also show that handedness in  $\mathbb{C}P^2$  counts e.g.  $T(-2, 5)$  is slice in  $\mathbb{C}P^2$  but  $g_s(T(2, 5)) = 1$  (see Theorem 1.2).

From now on,  $g_s(K)$  denotes the minimal genus over all isotopy classes of smooth connected oriented and compact surfaces whose boundary is the knot  $K \subset \partial(\mathbb{C}P^2 - \text{int}B^4)$ , and  $d$  denotes its corresponding degree in  $H_2(\mathbb{C}P^2 - \text{int}B^4, S^3, \mathbb{Z})$ . In **section 2.1**, we will prove Theorem 1.1 by explicitly giving a Seifert surface for  $T(-p, q)$  lying in  $\mathbb{C}P^2 - \text{int}B^4$  as stated in Claim 2.1.

**Theorem 1.1**  $g_s(T(-p, q)) \leq \frac{(q-1)(q-p-1)}{2}$ .

By an easy application of concordance theory, we can show that a slice knot in  $S^3$  is slice in  $\mathbb{C}P^2$ . However, the converse is not true since we can easily conclude from Theorem 1.1 that  $T(-p, p+1)$  ( $p \geq 2$ ) is slice in  $\mathbb{C}P^2$ .

A. Yasuhara [35] proved that there exist an infinite family  $T(-2, 2x_i + 1)$  which is non slice in  $\mathbb{C}P^2$ . However, the value of the smooth slice genus of any non-slice  $(\pm p, q)$ -torus knot in  $\mathbb{C}P^2$  is still unknown.

To answer this question, we will prove in **section 2.2** the following:

**Theorem 1.2 (Handedness)**

- (1)  $g_s(T(-2, 5)) = 0$  and  $g_s(T(2, 5)) = 1$ .
- (2)  $\frac{q \mp 1}{4} \leq g_s(T(2, q)) \leq \frac{q-3}{2}$  for  $q \equiv \pm 1 \pmod{4}$ .
- (3)  $g_s(T(-2, 7)) = 0$  with  $d = 4$ , and  $g_s(T(2, 7)) = 2$  with  $d \in \{0, \pm 1\}$ .

An interesting question is to find the degree and the smooth slice genus of torus knots in  $\mathbb{C}P^2$  in general. Note that  $T(p, q)$  is obtained from  $T(2, 3)$  by adding  $(p-1)(q-1) - 2$  half-twisted bands. This implies that there is a genus  $\frac{(p-1)(q-1)}{2} - 1$  concordance between  $T(2, 3)$  and  $T(p, q)$ . We claim that the smooth slice genus in  $\mathbb{C}P^2$  and the concordance genus are the same for any  $(p, q)$ -torus knot ( $0 < p < q$  and  $p$  and  $q$  are coprime). This let us hit to the following conjecture:

**Conjecture 1.1**  $g_s(T(p, q)) = \frac{(p-1)(q-1)}{2} - 1$ .

All known examples of slice torus knots in  $\mathbb{C}P^2$  are  $(-1)$ -twisted e.g.  $T(-p, 4p \pm 1)$  for any  $p \geq 2$  (see Figure 5). Notice that only left-handed torus knots can be slice in  $\mathbb{C}P^2$  with the right-handed trefoil as the only exception (see Figure 4). This can be proved by a using a theorem due to P. Gilmer and O. Ya Viro (see Theorem 2.2.1) and a theorem on non-positivity of the signatures of right-handed torus knots in general (see Ait Nouh- Yasuhara [4]). This let us meet with the following conjecture:

**Conjecture 1.2** A torus knot is slice in  $\mathbb{C}P^2$  if and only if it is  $(-1)$ -twisted.

In **section 3**, we disapprove the first Lawson's conjecture by proving the following:

**Proposition 3.1** Lawson's conjecture fails for either the pair  $(4, 1)$  or  $(4, -1)$  or  $(4, 0) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ .

In [5], we answer this conjecture by the positive for the small pairs  $(3, 3)$  and  $(6, 6)$ .

In **section 4**, we disapprove the second Lawson's conjecture [23] by proving Theorem 1.3.

Let  $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$  be the 4-manifold equipped with an elliptic fibration, and  $E(2) = E(1) \#_f E(1)$  be the fiber sum. We can check that  $E(2)$  is a  $K3$  surface and then  $b_2^+ = 3$  and  $b_2^- = 19$  (refer to R. Gompf and A. Stipsicz [15], pp.67 – 76 for more details on elliptic fibrations).

**Theorem 1.3** *There exist  $(a, b) \in H_2(E(2) \# E(2)) = H_2(E(2)) \oplus H_2(E(2))$  such that  $a \cdot a \geq 0$  and  $b \cdot b \geq 0$ , and the genus of  $a$  (resp.  $b$ ) is minimal in  $H_2(E(2))$  (resp.  $H_2(E(2))$ ), but the genus of  $a + b$  is less and not equal to the sum of the genus of  $a$  and the genus of  $b$ .*

The genus function  $G$  is defined on  $H_2(X, \mathbb{Z})$  as follows: For  $\alpha \in H_2(X, \mathbb{Z})$ , consider

$$G(\alpha) = \min\{\text{genus}(\Sigma) \mid \Sigma \subset X \text{ represents } \alpha, \text{ i.e., } [\Sigma] = \alpha\}$$

Where  $\Sigma$  ranges over closed, connected, oriented surfaces smoothly embedded in the 4-manifold  $X$ . Note that  $G(-\alpha) = G(\alpha)$  and  $G(\alpha) \geq 0$  for all  $\alpha \in H_2(X, \mathbb{Z})$  (An excellent reference is Gompf-Stipsicz [14]).

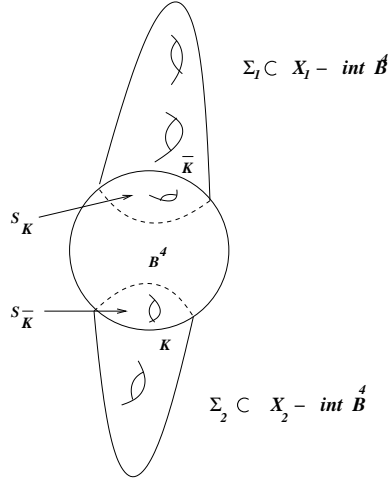


Figure 2: The gluing of surfaces technique

In our setting, the connection between knot theory and dimension four topology is based on the following construction depicted in Figure 2: Let  $K$  be a knot in  $S^3$ , then the dual knot of  $K$  is the inverse of the mirror-image  $K^*$  of  $K$  i.e.  $\bar{K} = -K^*$  ([16]). Denote by  $X_1^4$  and  $X_2^4$  two oriented and closed 4-manifolds and let  $(\Sigma_i, \partial\Sigma_i) \subset (X_i - \text{int}B^4, S^3)$  for  $i = 1, 2$  two compact and oriented surfaces such that  $\partial\Sigma_1 = K$  and  $\partial\Sigma_2 = \bar{K}$ . Denote by  $\Sigma'_1 = \Sigma_1 \bigcup_K S_K$  and  $\Sigma'_2 = \Sigma_2 \bigcup_K S_{\bar{K}}$  where  $S_K$  (resp.  $S_{\bar{K}}$ ) is the standard Seifert surface for  $K$  (resp.  $\bar{K}$ ) in  $B^4$ . Gluing  $\Sigma'_1$  and  $\Sigma'_2$  along their boundaries yields a new closed surface  $\Sigma'_1 \bigcup_K \Sigma'_2$  such that  $[\Sigma'_1 \bigcup_K \Sigma'_2] = [\Sigma'_1] + [\Sigma'_2] \in H_2(X_1 \# X_2, \mathbb{Z})$  and  $g(\Sigma'_1 \bigcup_K \Sigma'_2) = g(\Sigma_1) + g(\Sigma_2) - g_4(K) - g_4(\bar{K})$ , where  $g_4(K)$  denotes the 4-ball genus of  $K$ .

Let  $a = [\Sigma'_1] = [\Sigma_1 \bigcup_K S_K] \in H_2(X_1, \mathbb{Z})$  and  $b = [\Sigma'_2] = [\Sigma_2 \bigcup_K S_{\bar{K}}] \in H_2(X_2, \mathbb{Z})$ . Then  $a + b = [\Sigma_1 \bigcup_K \Sigma_2]$ . It is important to notice here that under the assumptions  $g_4(K) \geq 1$ , and  $a$  and  $b$  are minimal, then  $G(a + b) < G(a) + G(b)$ . Indeed,  $\Sigma_1 \bigcup_K \Sigma_2$  skips the four ball genus of  $K$  and  $\bar{K}$ . In this fashion, we will present a counterexample to the second Lawson's conjecture as stated in Theorem 1.3 and illustrated in Figure 8 of page 12 with  $K = 4_1$  and  $X_1 = X_2 = E(2)$  in which we find  $a$  and  $b$  as described above such that  $G(a + b) < G(a) + G(b)$ .

If we take the standard connected sum of  $\Sigma_1$  and  $\Sigma_2$ , then this does not affect the genus. More precisely, we will get a new surface  $\Sigma_1 \# \Sigma_2$  whose genus is the sum of the genus of  $\Sigma_1$  and  $\Sigma_2$ . This proves that if  $X = X_1 \# X_2$  is the connected sum of two closed 4-manifolds, and if  $(a, b) \in H_2(X) = H_2(X_1) \oplus H_2(X_2)$

then  $G(a + b) \leq G(a) + G(b)$ . However, the inequality can be strict. Therefore, the minimal genus in a connected sum of 4-manifolds is not always the sum of the minimal genus in each factor.

We mention here that G. Mikhalkin ([28]) has shown that the genus-minimizing surfaces in  $\mathbb{C}P^2$  can have their genus reduced further after direct sum with additional copies of  $\mathbb{C}P^2$  i.e.  $\mathbb{C}P^2 \# \dots \# \mathbb{C}P^2$ .

So far, there is no theory for 4-manifolds with even  $b_2^+$ , and Seiberg-Witten theory applies mainly to 4-manifolds with odd  $b_2^+ > 1$ . Connected sums of 4-manifolds with even  $b_2^+$  is an open area of research where gauge theory remains inefficient. In light of the above techniques, we treat  $\mathbb{C}P^2 \# \mathbb{C}P^2$  and  $E(2) \# E(2)$ .

## 2 Proof of statements

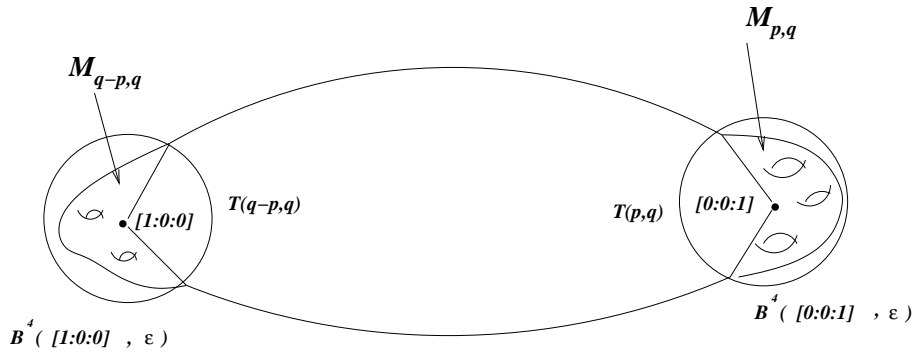


Figure 3: The surfaces  $\Sigma$  and  $\tilde{\Sigma}$

### 2.1 Smooth Seifert surface spanning a $(-p, q)$ -torus knot in $\mathbb{C}P^2$

To prove Theorem 1.1, we explicitly give a smooth complex Seifert surface for  $T(-p, q)$ , and find its genus (see Claim 2.1). Recall some preliminaries: In homogeneous coordinates  $[x : y : z]$  where  $(x, y, z) \in \mathbb{C}^3$ , the complex projective plane  $\mathbb{C}P^2$  is covered by three affine charts  $U_x := \{[1 : y : z] \in \mathbb{C}P^2 \mid (y, z) \in \mathbb{C}^2\}$ , and  $U_y := \{[x : 1 : z] \in \mathbb{C}P^2 \mid (x, z) \in \mathbb{C}^2\}$  and  $U_z := \{[x : y : 1] \in \mathbb{C}P^2 \mid (x, y) \in \mathbb{C}^2\}$ . Let  $\Sigma$  be the curve in  $\mathbb{C}P^2$  that is given in homogeneous coordinates by  $x^p z^{q-p} + y^q = 0$  ( $0 < p < q$ ;  $p$  and  $q$  are coprime). This curve has two singularities: the one in  $U_z$  at  $[x : y : z] = [0 : 0 : 1]$  whose link is  $T(p, q)$ , and the other one in  $U_x$  at  $[1 : 0 : 0]$  whose link is  $T(q - p, q)$  (see Figure 3). Thus the intersection number with the  $\mathbb{C}P^1$  ( $y = 0$ ) is  $p + (q - p) = q$  as required. Since  $\Sigma$  has degree  $q$ , we can desingularize it by perturbing its equation

to obtain a smooth curve  $\tilde{\Sigma}$ . By Thom's conjecture, that is proved by P. Kronheimer and T. Mrowka (see [20]), the genus of  $\tilde{\Sigma}$  is  $(q-1)(q-2)/2$ .

**Claim 2.1**  $M_{p,q}^\infty = \tilde{\Sigma} \cap (\mathbb{C}P^2 - \text{int}(B^4([0:0:1], \epsilon)))$  (see Figure 3) is a smooth complex Seifert surface for  $T(-p, q)$  in  $\mathbb{C}P^2$  whose genus is  $\frac{(q-1)(q-2)}{2} - \frac{(p-1)(q-1)}{2}$ .

**Proof** Desingularizing the singularity  $[0:0:1]$  (resp.  $[1:0:0]$ ) replaces the cone on  $T(p, q)$  (resp.  $T(q-p, q)$ ) by its Milnor fiber  $M_{p,q}$  (resp.  $M_{q-p,q}$ ), which is the obvious Seifert surface for the torus knot  $T(p, q)$  whose genus is  $(p-1)(q-1)/2$  (resp.  $(q-p-1)(q-1)/2$ ) ( see [21],[6]). Thus, if we undo the perturbation to recover  $\Sigma$ , we must subtract such a term for each singularity: the genus of  $\Sigma$  is then  $(q-1)(q-2)/2 - (p-1)(q-1)/2 - (q-p-1)(q-1)/2 = 0$ . Thus,  $\Sigma$  is a sphere with two locally knotted points. Since  $\tilde{\Sigma} = M_{p,q}^\infty \bigcup_{T(p,q)} M_{p,q}$ , then  $\partial M_{p,q}^\infty = M_{p,q} \cap S^3([0:0:1], \epsilon)$ . Therefore  $M_{p,q}^\infty$  bounds  $T(-p, q)$ , and  $M_{p,q}^\infty$  is smooth, complex and compact. In addition,  $g(M_{p,q}^\infty) = g(\tilde{\Sigma}) - g(M_{p,q})$ , or equivalently  $g(M_{p,q}^\infty) = (q-1)(q-2)/2 - (p-1)(q-1)/2$ .

**Proof of Theorem 1.1** The proof is an immediate corollary of Claim 2.1.

**Remark** Notice that the degree  $d$  of a genus-minimizing Seifert surface for  $T(-p, q)$  is different from  $q$  in general. Indeed,  $T(-p, 4p \pm 1)$  is slice with  $d = 2p$  ( $q = 4p \pm 1$ ) (see Proposition 1.1). Thus the relative Thom conjecture is false in general.

## 2.2. Proof of Theorem 1.2

We need some preliminaries derived from old gauge theory:

**Theorem 2.2.1** (P. Gilmer and O. Ya. Viro [12], [33]) Let  $X$  be an oriented, compact 4-manifold with  $\partial X = S^3$ , and  $K$  a knot in  $\partial X$ . Suppose  $K$  bounds a surface of genus  $g$  in  $X$  representing  $\xi \in H_2(X, \partial X)$ .

(1) If  $\xi$  is divisible by an odd prime  $d$ , then:  $|\frac{d^2-1}{2d^2}\xi^2 - \sigma(X) - \sigma_d(k)| \leq \dim H_2(X; \mathbb{Z}_d) + 2g$ .

(2) If  $\xi$  is divisible by 2, then:  $|\frac{\xi^2}{2} - \sigma(X) - \sigma(k)| \leq \dim H_2(X; \mathbb{Z}_2) + 2g$ .

In the following, let  $b_2^+$  (resp.  $b_2^-$ ) denotes the dimension of the maximal positive (resp. negative) subspace for the intersection form on  $H_2(X, \mathbb{Z})$ .

**Theorem 2.2.2** (K. Kikuchi [18]) Let  $X$  be a closed, oriented and smooth 4-manifold such that: (1)  $H_1(X)$  has no 2-torsion; and (2)  $b_2^{\pm 1} \leq 3$ .

If  $\xi$  is a characteristic class of  $H_2(X, \mathbb{Z})$  represented by an embedded 2-sphere in  $X$ , then:  $\xi^2 = \sigma(X)$

**Theorem 2.2.3** (D. Acosta [1], R. Fintushel [10], A. Yasuhara [35]) Let  $X$  be a smooth closed oriented simply connected 4-manifold with  $m = \min(b_2^+(X), b_2^-(X))$  and  $M = \max(b_2^+(X), b_2^-(X))$ , and assume that  $m \geq 2$ . Suppose  $\Sigma$  is an embedded surface in  $X$  of genus  $g$  so that  $[\Sigma]$  is characteristic. Then

$$g \geq \begin{cases} \frac{|\Sigma \cdot \Sigma - \sigma(X)|}{8} + 2 - M & \text{if } \Sigma \cdot \Sigma \leq \sigma(X) \leq 0 \text{ or } 0 \leq \sigma(X) \leq \Sigma \cdot \Sigma \\ \frac{9(|\Sigma \cdot \Sigma - \sigma(X)|)}{8} + 2 - M & \text{if } \sigma(X) \leq \Sigma \cdot \Sigma \leq 0 \text{ or } 0 \leq \Sigma \cdot \Sigma \leq \sigma(X) \\ \frac{|\Sigma \cdot \Sigma - \sigma(X)|}{8} + 2 - m & \text{if } \sigma(X) \leq 0 \leq \Sigma \cdot \Sigma \text{ or } \Sigma \cdot \Sigma \leq 0 \leq \sigma(X) \end{cases}$$

To prove Theorem 1.2., we need the following:

**Corollary 2.1**  $T(2, 5)$  is not slice in  $\mathbb{C}P^2$

**Proof** Assume for a contradiction that  $T(2, 5)$  is slice in  $\mathbb{C}P^2$ , then there exist a properly embedded disk  $\Delta \subset \mathbb{C}P^2 - \text{int}B^4 = M_1$  such that  $\partial\Delta = T(2, 5)$ . Let  $[\Delta] = d\gamma$ , where  $\gamma$  is the standard generator of  $H_2(\mathbb{C}P^2 - \text{int}B^4, S^3, \mathbb{Z})$ . If  $d$  is even, then by Theorem 2.2.1,  $|\frac{d^2}{2} - \sigma(T(2, 5)) - 1| \leq 1$ . By A.G. Tristram [32],  $\sigma(T(2, 5)) = -4$ , and then  $d$  satisfies  $d^2 + 3 \leq 1$ , a contradiction.

Assume now that  $d$  is odd. We can check that  $T(-2, 5)$  is obtained from the unknot  $T(-2, 1)$  by a single  $(-2, 2)$ -twisting. In [25] and [9], the authors proved using Kirby's calculus on the Hopf link [19], that there exist  $D \subset M_1 \# M_2 = S^2 \times S^2 - \text{int}B^4 = M_2$  such that  $[D] = 2\alpha + 2\beta$  and  $\partial D = T(-2, 5)$ . The sphere  $[\Delta \cup D] = d\gamma + 2\alpha + 2\beta \in \mathbb{C}P^2 \# S^2 \times S^2$  is a characteristic class. By Kikuchi's Theorem,  $[S^2] \cdot [S^2] = \sigma(M^4)$  and then  $d^2 + 8 = 1$ , a contradiction.

**Proof of Theorem 1.2**

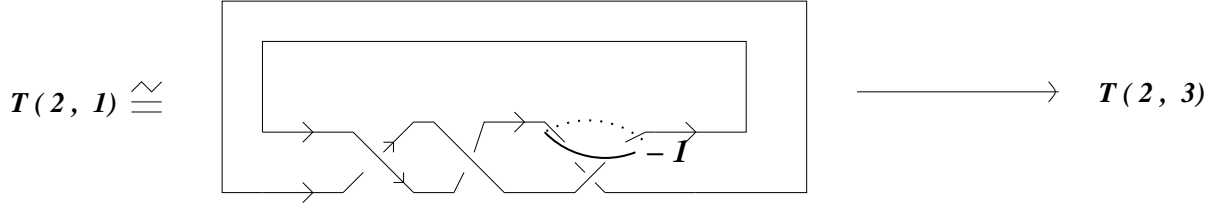


Figure 4:

Notice first that  $T(2, 3)$  is obtained from the unknot by  $(-1, 0)$ -twisting (see Figure 4), which implies that  $T(2, 3)$  is smoothly slice in  $\mathbb{C}P^2$ .

(1) J. Song and H. Goda and C. Hayashi proved in [13] that  $T(2, 5)$  is obtained from the unknot by a single  $(+1, 3)$ -twisting. Therefore,  $T(-2, 5)$  is obtained from the unknot by a single  $(-1, 3)$ -twisting ([2]). From [9] and [25] we deduce that  $T(-2, 5)$  is slice in  $\mathbb{C}P^2$ . Notice that  $T(2, 5)$  is obtained from  $T(2, 3)$  by adding two bands. Thus there is a genus-one cobordism between  $T(2, 3)$  and  $T(2, 5)$ , and therefore  $g_s(T(2, 5)) \leq 1$ . Corollary 2.1 yields that  $g_s(T(2, 5)) = 1$ .

(2) Assume that  $q = 4n \pm 1$  for some integer  $n \geq 1$ , and prove that  $\frac{q \mp 1}{4} \leq g_s(T(2, q)) \leq \frac{q-3}{2}$ .

**Case 1**  $q = 4n + 1$  for some integer  $n \geq 1$ :

Let  $\Sigma_g \subset \mathbb{C}P^2 - \text{int}B^4$  be a genus-minimizing  $g$  surface such that  $\partial\Sigma_g = T(2, 4n+1)$  with  $[\Sigma_g] = d\gamma$  where  $\gamma$  is the standard generator of  $H_2(\mathbb{C}P^2, \mathbb{Z})$ . Note that  $T(-2, 4n+1)$  is obtained from  $T(-2, 1)$  by a single  $(-2n, 2)$ -twisting. By [9] and [25], there exist a disk  $(D, \partial D) \subset (S^2 \times S^2 - \text{int}B^4, S^3)$  such that  $\partial D = T(-2, 4n+1)$  and  $[D] = 2\alpha + 2n\beta \in H_2(S^2 \times S^2 - \text{int}B^4, S^3)$ . The surface  $\Sigma = \Sigma_g \cup D \subset \mathbb{C}P^2 \# S^2 \times S^2$  satisfies  $[\Sigma] = d\gamma + 2\alpha + 2n\beta \in H_2(\mathbb{C}P^2 \# S^2 \times S^2)$ . Thus  $[\Sigma]^2 = d^2 + 8n$ , so blowing up  $\Sigma \subset \mathbb{C}P^2 \# S^2 \times S^2$  a number of times equal to  $d^2 + 8n$  gives a genus  $g$  surface  $\tilde{\Sigma} \subset \mathbb{C}P^2 \# S^2 \times S^2 \# (d^2 + 8n)\overline{\mathbb{C}P^2}$  (the proper transform) with  $[\tilde{\Sigma}]^2 = 0$ . If  $e_i$  denotes the homology class of the exceptional sphere in the  $i^{\text{th}}$  blow-up ( $i = 1, 2, \dots, d^2 + 8n$ ), then  $[\tilde{\Sigma}] = d\gamma + 2\alpha + 4\beta - \sum_{i=1}^{d^2+8n} e_i$ .

If  $d$  is odd then  $X = \mathbb{C}P^2 \# S^2 \times S^2 \# (d^2 + 8n)\overline{\mathbb{C}P^2}$  has a signature  $\sigma(X) = 1 - d^2 - 8n$ . The last inequality of Theorem 2.2.3, yields that  $g \geq \frac{8n + d^2 - 1}{8}$  (\*), which implies that  $g \geq n$ .

If  $d$  is even, then Gilmer-Viro's Theorem 2.2.1 implies that  $|\frac{d^2}{2} - 1 - \sigma(T(2, 4n + 1))| \leq 1 + 2g$ . Since  $\sigma(T(2, 4n + 1)) = -4n$  (see Tristram [32]), then  $|\frac{d^2}{2} - 1 + 4n| \leq 1 + 2g$ , which implies that  $2n - 1 \leq g$ , and therefore  $n \leq g$ . Therefore if  $q = 4n + 1$ , then  $\frac{q-1}{4} \leq g$ .

It is not hard to prove that  $g \leq \frac{q-3}{2}$  by induction. Indeed,  $T(2, 3)$  is slice in  $\mathbb{C}P^2$  and there is a genus-two cobordism between  $T(2, q)$  and  $T(2, q + 2)$  and therefore, there is a genus  $\frac{q-3}{2}$  between  $T(2, 3)$  and  $T(2, q)$ .

**Case 2** If  $q = 4n - 1$  then the proof is similar to Case 1, and we get  $\frac{q+1}{4} \leq g \leq \frac{q-3}{2}$

(3) We can deduce from Proposition 1.1, whose proof follows, that  $T(-2, 7)$  is slice in  $\mathbb{C}P^2$  with  $d = 4$ . Since  $T(2, 7)$  is obtained from  $T(2, 3)$ , which is slice in  $\mathbb{C}P^2$ , by adding four half-twisted bands, then  $g(T(2, 7)) \leq 2$ . Assume first that  $d$  is odd, then letting  $n = 2$  in the inequality  $(\star)$  yields that  $g_s(T(2, 7)) = 2$  and  $d = \pm 1$ . If  $d$  is even, then Gilmer-Viro's Theorem 2.2.1 implies that if  $g_s(T(2, 7)) = 2$  then  $d = 0$ . Therefore  $d \in \{0, \pm 1\}$ .

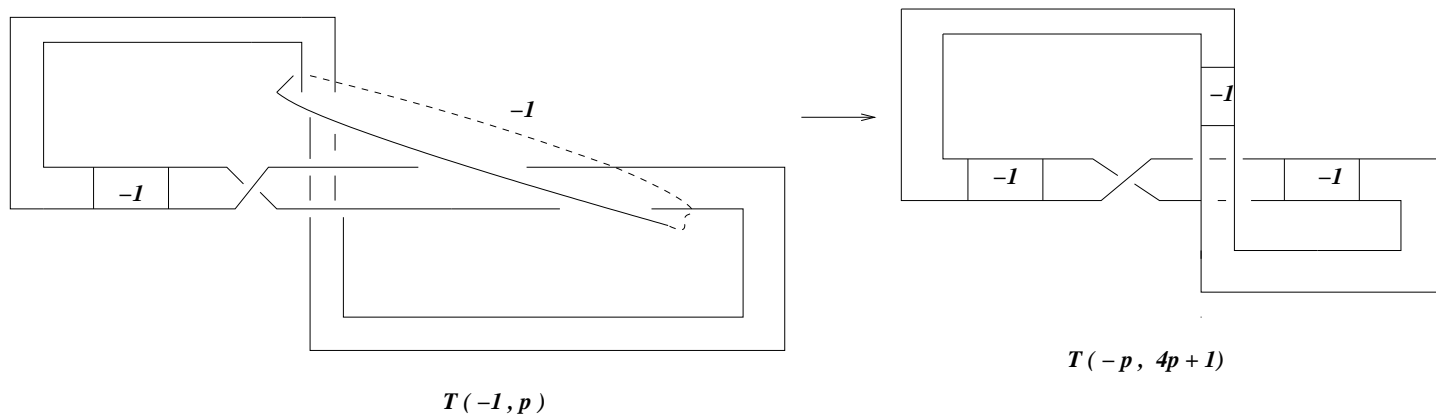


Figure 5:

### Proof of Proposition 1.1

**Proposition 1.1**  $T(-p, 4p \pm 1)$  for  $p \geq 2$  is slice in  $\mathbb{C}P^2$ .

**Proof** The movie described in Figure 5 proves that  $T(-p, 4p + 1)$  is obtained from  $T(-1, p)$  by a single  $(-1, 2p)$ -twisting. The proof is similar for  $T(-p, 4p - 1)$  provided that we start from  $T(1, p)$ .

### 3 Minimal genus problem in $\mathbb{C}P^2\#\mathbb{C}P^2$

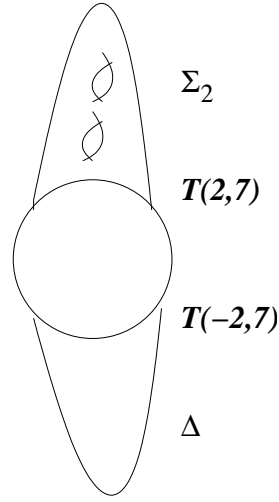


Figure 6:

T. Lawson conjectured in [23] that the minimal genus of  $(m, n) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2)$  is given by  $\binom{|m|-1}{2} + \binom{|n|-1}{2}$  -this is the genus realized by the connected sum of algebraic curves in each factor. In [5], we answer this conjecture by the positive for the small pairs (3, 3) and (6, 6). The proofs use twisting of knots in  $S^3$  and gauge theory. We answer here this conjecture by the negative in general.

**Proposition 3.1** Lawson's conjecture fails for either the pair (4, 1) or (4, -1) or (4, 0)  $\in H_2(\mathbb{C}P^2\#\mathbb{C}P^2)$ .

**Proof** By Proposition 1.1, we deduce that  $T(-2, 7)$  is slice in  $\mathbb{C}P^2$  with degree  $d = 4$ . Therefore, there exist a smooth disk  $(\Delta, \partial\Delta) \subset (\mathbb{C}P^2 - \text{int}B^4, S^3)$  such that  $\partial\Delta = T(-2, 7)$  and  $[\Delta] = 4\gamma$ , where  $\gamma$  is the standard generator of  $H_2(\mathbb{C}P^2 - \text{int}B^4, S^3)$ . By Theorem 1.2, the smooth slice genus of  $T(2, 7)$  in  $\mathbb{C}P^2$  is two. Thus, there exist a genus-two surface  $(\Sigma_2, \partial\Sigma_2) \subset (\mathbb{C}P^2 - \text{int}B^4, S^3)$  such that  $\partial\Sigma_2 = T(2, 7)$  and  $[\Sigma_2] = d\gamma \in H_2(\mathbb{C}P^2, \mathbb{Z})$  where  $d \in \{0, \pm 1\}$ . By Theorem 1.2, the genus-two smooth surface  $\Sigma = \Delta \cup \Sigma_2$  in  $\mathbb{C}P^2\#\mathbb{C}P^2$  satisfies  $[\Sigma] = 4\gamma_1 + d\gamma_2 \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})$  with  $d \in \{0, \pm 1\}$  (see Figure 6). If Lawson's conjecture were true, then the genus of  $\Sigma$  which is two should be greater or equal to the proposed Lawson's minimal genus for the pair  $(4, d) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})$  which is  $3 + \frac{(|d|-1)(|d|-2)}{2}$  where  $d \in \{0, \pm 1\}$ , a contradiction.

#### 4 Minimal genus problem of connected sum of symplectic surfaces

Let  $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$  be the 4-manifold equipped with an elliptic fibration, and let  $F$  be a regular fiber of  $E(1)$ . Then a tubular neighborhood of  $F$  is  $\nu(F) \cong D^2 \times T^2$ , and therefore  $\partial\nu(F) = T^3 (= \partial(E(1) - \nu(F)))$ . Define  $E(2) = (E(1) - \nu(F)) \cup_{T^3} (E(1) - \nu(F))$ , or simply  $E(2) = E(1) \#_F E(1)$  which is called the fiber sum.  $E(2)$  is a  $K3$  surface and then  $b_2^+ = 3$  and  $b_2^- = 19$ . We have  $H_2(E(2), \mathbb{Z}) \cong \mathbb{Z}^{22}$ , and a basis is given by 16 spheres  $\{S_1, \dots, S_{16}\}$  of square  $-2$ , realizing  $-2E_8$ , and three  $K3$ -nucli  $N_i(2) = N(\sigma_i \cup T_i) (i = 1, 2, 3)$  which can be endowed with a symplectic structure, and such that the intersection matrix of  $(\sigma_i, T_i) (i = 1, 2, 3)$  is  $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$  ([15], p. 72).

**Claim 4.1** The intersection matrix of  $(\sigma, T, \sigma + 3T)$  is  $\begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{pmatrix}$ .

**Proof** By resolving the singular points,  $\sigma + 3T$  is a genus three surface. Since  $\sigma^2 = -2$ ,  $\sigma T = 1$  and  $T^2 = 0$  then  $(\sigma + 3T)^2 = \sigma^2 + 6\sigma T + T^2 = 4$ , and  $\sigma(\sigma + 3T) = \sigma^2 + 3\sigma T = 1$ .

#### *Proof of Theorem 1.3*

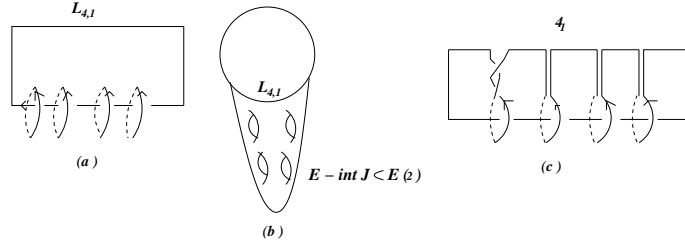


Figure 7:

**Claim 4.2** Represent  $\sigma + 3T$  by three disjoint copies of the fiber denoted respectively by  $T_2, T_3$  and  $T_4$ . For convenience, we denote  $T = T_1$ . There exist a surface  $E \subset E(2) - \text{int}(B^4)$  such that:

- $\partial(E - J) = E \cap \partial J = L_{4,1}$  where the  $(4, 1)$ -torus link  $L_{4,1}$  is depicted in Figure 7(a), and
- $[E - J] = [\sigma] + [T_1] + [T_2] + [T_3] + [T_4]$  in  $H_2(E(2) - \text{int}(B^4), S^3, \mathbb{Z})$ .

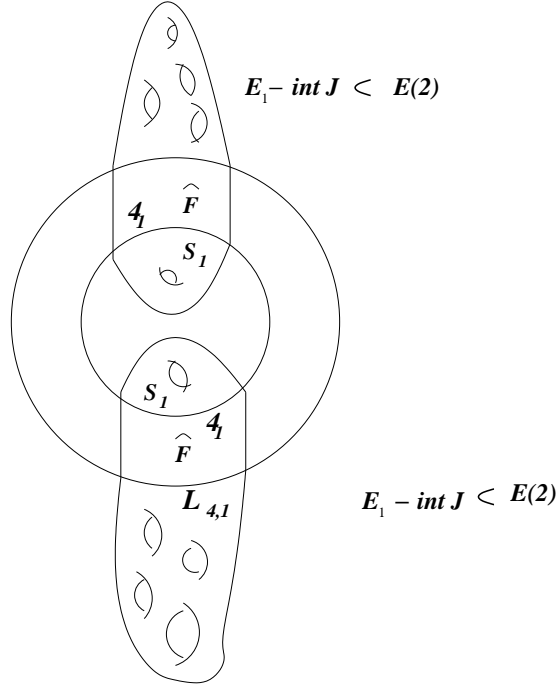


Figure 8:

**Proof** Consider  $E = (\frac{2}{7}, \frac{2}{7}) \times \sigma \cup \bigcup T_1 \times (\frac{3}{7}, \frac{3}{7}) \cup T_2 \times (\frac{4}{7}, \frac{4}{7}) \cup T_3 \times (\frac{5}{7}, \frac{5}{7}) \cup T_4 \times (\frac{6}{7}, \frac{6}{7})$ , and the 4-ball  $J = [\frac{1}{7}, \frac{6}{7}]^2 \times [\frac{1}{7}, \frac{6}{7}]^2$ .

**Proof of Theorem 1.3**

Notice that the figure eight  $4_1$  knot is both amphicheiral and invertible, and then  $4_1 \cong \overline{4_1}$ , where  $\overline{4_1}$  is the dual knot of  $4_1$ . By Claim 4.2, there exist a surface  $E$  and a 4-ball  $J$ , such that:  $\partial(E - J) = L_{4,1}$  (see Figure 7(b)). Since  $4_1$  is obtained from  $L_{4,1}$  by fusion (see Figure 7(c)), then there exist a 6-punctured sphere  $\hat{F}$  in  $S^3 \times [0, 1] \subset J$  such that we can identify this band surgery with  $\hat{F} \cap (S^3 \times \{1/2\})$ , and  $\partial\hat{F} = L_{4,1} \cup 4_1$  with  $L_{4,1}$  lies in  $S^3 \times \{0\} \cong \partial J \times \{0\}$  and  $4_1$  lies in  $S^3 \times \{1\} \cong \partial J \times \{1\}$ . By Schönflies theorem [31],  $S^3 \times \{1\} (\cong \partial J \times \{1\})$  bounds a 4-ball  $B^4 \subset J$ . Let  $(S_1, \partial S_1) \subset (int B^4, \partial B^4)$  be a genus one Seifert surface for  $4_1$  ( $g_4(4_1) = 1$ ), then  $\Sigma_1 = (E - int(J)) \bigcup \hat{F} \bigcup S_1$  is represented by  $a = [\sigma] + [T] + [\sigma + 3T]$ . Since the genus of  $E - int(J)$  is four, then the genus of  $\Sigma_1$  is five. Since the  $K_3$ -nucleus is symplectic, then by the adjunction formula  $1 + \frac{[\Sigma_1] \cdot [\Sigma_1]}{2} = 1 + \frac{8}{2} (= 5)$  (Ozsváth-Szabo [29]). This implies that  $a = [\Sigma_1] \in H_2(E(2), \mathbb{Z})$  is genus-minimizing in its homology class. Let  $\Sigma_2$  be another copy of  $\Sigma_1$  in  $E(2)$ , and denote  $[\Sigma_2] = b$  ( $= a$ ). Notice that  $a \cdot a = b \cdot b = 8$ , and that  $[\Sigma_1 \bigcup_{4_1} \Sigma_2] = a + b \in H_2(E(2) \# E(2), \mathbb{Z})$ . Therefore the genus of the class  $a + b = 2([\sigma] + [T] + [\sigma + 3T])$  is 8 (which is the genus of  $\Sigma_1 \bigcup_{4_1} \Sigma_2$ ). If the second Lawson's conjecture were true, then the homology class of  $a + b$  would have genus  $5 + 5 = 10$ ; a contradiction.

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