Covariation and Functions: Developing Mathematical Knowledge for Teaching in Pre-service Secondary Teachers[1]

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Abstract: The purpose of this paper is to present results from of an ongoing design research study that explores the development of mathematical knowledge for teaching (MKT) in prospective secondary teachers. The undergraduate math majors participated in this study while enrolled in mathematics courses designed to 1) strengthen their own understanding of collegiate mathematics through inquiry, 2) enhance their understanding of the strategies and models that children use in mathematics problem solving, and 3) develop their capacity to be able to see the development of the big mathematical ideas in children's work and to link this understanding to related big ideas in their own investigations. The paper examines in detail their work related to covariation and function in three iterations of the course and the roles of these critical topics in MKT related to understanding of children's thinking are discussed.

Section 1. Introduction

Research on the development of understanding of covariation function has proliferated over the past two decades, in part due to the calculus reform efforts during the 1990s and in part to attempts to introduce algebra earlier into the curriculum in middle and elementary school. Developmental frameworks have been proposed in multiple contexts, for example in functions (Monk 1987, Thompson, 1991, Dubinsky&Harel, 1992), in differential equations (Rasmussen, 2001), or in modeling (Carlson et al, 2002). The research described here considers the understandings of function and covariation of undergraduate math majors in relation to their ability to recognize similar emergent big ideas, strategies, and models in the work of children in grades 4-8. These undergraduates, as pre-service teachers, were enrolled in a mathematics course (Math 181A at UCSB) designed to foster early Mathematical Knowledge for Teaching (MKT) with a focus on number sense and early algebra, in particular proportional reasoning, functions and covariation. The course combined problem solving (collegiate level) with case studies of children in grades 4-8. The problem solving and the case studies were matched so that the undergraduates could search for emergent big ideas, strategies and models in the children's work and link what they found to their own problem solving. The hypothesis was that this linkage would heighten the undergraduates' ability to identify developing ideas in children and thereby promote early development of components of MKT.

In recent work (Jacob, et al 2009) it was noted that components of the hypothesized process appears to work, although non math majors preparing for elementary teaching

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were better able than math majors to identify the development of big ideas in children's work (which involved analysis of fourth grade work) after participating in the same instructional sequence. The paper noted a relationship between undergraduates' approaches to problem solving, the ideas they uncovered in their own work, and the ideas they were able to identify in children's work. Furthermore, those undergraduates whose problem solving didn't allow them to confront key obstacles (often because they or a partner had a quick solution) appear to be disadvantaged when asked to analyze children's work. This and similar observations by other researchers suggest that the pre-service education of secondary teachers will be enhanced if they have to grapple with problems designed to raise obstacles similar to those their future students may face.

In Math 181A (and in a similar course for pre-service elementary teachers) the metaphor "landscape" is employed following the work of Fosnot and Dolk (2002ab, 2003), where big ideas, strategies, and models used by children are displayed graphically. The components that are usually precursors to others placed lower on the landscape. The metaphor deliberately evokes nonlinear features of development, aligned with an instructional objective to move students towards the horizon according to their development. Fosnot and Dolk developed their landscapes through their (and other's) developmental research on children's thinking. The research presented here analyzes Math 181A undergraduates' problem solving in order to develop landscapes of their understanding of covariation and function and to link these landscapes with their abilities to make sense of children's work. Our belief is that such landscapes can guide hypothetical trajectories for content and pedagogy courses.

This paper focuses on emergent big ideas linking covariation and function in the development of undergraduate math majors in relation to similar ideas in children's developing number sense and early algebra. Schifter and Fosnot (1993) describe *big ideas* as "the central, organizing ideas of mathematics—principles that define mathematical order" and as noted by Fosnot and Dolk (2001) "as such, they are deeply connected to the structures of mathematics. They are, however, also characteristic of learner's shifts in learner's reasoning—shifts in perspective, in logic, in the mathematical relationships they set up." This latter characteristic of big ideas drives the research in this paper. Big ideas are not theorems or topics (which are difficult points for our pre-service teachers). Instead we can identify their emergence when (1) a learner's approaches and behaviors while engaged in inquiry change substantially and (2) these changes are based upon important mathematical structures and principles. In this research we endeavor to capture these important transitions, the contexts that appeared to facilitate their emergence (or reemergence), and place them along with relevant strategies and models on a landscape. Then we link their emergence (or lack of) to these same undergraduates abilities to make sense of related ideas in children's work.

The third book of Fosnot and Dolk (2003) is used as the text for Math 181A, and their notions of big ideas, strategies, and models are explicitly examined in the course. A key component, then, of the class is to help undergraduates develop their abilities to find evidence of emergent big ideas, strategies, and models in children's work. To develop these abilities case studies of children doing mathematics are studied. For this we used several CDROM of Dolk and Fosnot CDROM (2005) and case studies from Smith et al (2005). Then, the work of the children was discussed in relation to the undergraduates problem solving. Math 181A students write three papers per quarter linking their own problem solving to work of children in the cases. Two such writing assignments are examined here, one where they write about children's work in 7th grade considering figurate numbers, and a second where they wrote about 4th grade students considering two division contexts. In both, the children use ideas related to covariation and function in their inquiry, and our interest is the undergraduates' abilities to locate and describe the use of these ideas.

The Instructional Sequences, Research Design and Artifacts Considered.

The paper considers work of pre-service teachers in three iterations of Math 181A, where the hypothetical learning trajectories and class investigations were modified the second and third times based upon formative evaluation. Three in-class investigations designed to develop both mathematics and MKT in the area of covariation and function were analyzed, and provided evidence for the initial big ideas located on the landscape. The undergraduates' work on these investigations was collaborative. The three authors coded their written work, video of group discussions during the investigation, and video and posters from class presentations and discussions of the investigations (referred to as a class "congress" using the terminology of Fosnot and Dolk, where the instructors attempt to scaffold development of the ideas through their choice of presenters). This work is described in Section 3 and provides the basis for construction of the Covariation and Function Landscape presented in Section 5. Following this, a written assignment from each class was analyzed in order to assess undergraduates grasp of these ideas and to examine the relationship between how they access these ideas and their abilities to identify related big ideas in children's work. This is discussed in Section 4. As noted in (Jacob, et. al, 2009) these undergraduates proved to be quite capable of identifying children's strategies and in "retelling" what they did, but their ability to identify children's shifts in thinking as exemplified by the emergence of a big idea is much more problematic. Consequently, the focus here is on big ideas.

The initial learning trajectory in the first version of Math 181A (N = 29) included an investigation of covariation in proportional reasoning called the "Eudoxus problem", followed by examination of a 7th grade class' work and discussion of a figurate number problem, the Case of Ed Taylor (Smith et al, 2005). The written assignment, approximately seven weeks into the course, linking the Eudoxus investigation and use of ideas of covariation and function in children's work in the case of Ed Taylor provided the basis for formative evaluation and a modified hypothetical learning trajectory for the second iteration of the course. Up to this point the course included collegiate level investigations involving error correcting codes, power series solutions to ordinary differential equations, and problems involving elementary number theory, as well as opportunities to analyze children's thinking in early number sense, including video cases related to multiplication, division, and fractions in grades 3-5.

After noting the difficulty the undergraduates had in describing the relationship between covariation and function, the revised learning trajectory for the second teaching of Math 181A (N = 22) included more opportunities to develop these topics, including homework involving proof by induction and two new investigations set in a continuous context. The hypothesis was that as mathematics majors, the undergraduates might find continuous (calculus related) contexts more appealing and easier to make sense of, than simply looking at middle school problems where they tended to trivialize the mathematics. The first investigation was based on Cavalieri's principle except in reverse (instead of calculating volume by integrating area of slices, they were to interpret rates of change of volume as areas of slices.) The second investigation required that they construct an ordinary differential equation as an example of covariation and that they examine the relationship between an iterative solution (Euler's method) and a continuous solution obtained by separation of variables. These investigations proved to be problematic, indeed difficult, and as a result enabled us to expand the landscape of function and covariation. The expanded instructional sequence (including the Eudoxus problem, both continuous problems, and the Ed Taylor case) provided further evidence for development of a Covariation Function landscape.

In order to investigate their abilities to make sense of children's mathematical thinking in relation to covariation and function, a new writing prompt was designed for the second two iterations of Math 181A. The undergraduates wrote individual papers (at week four during the second iteration) comparing the role of covariation and function on a new calculus level problem (Frank and Doris' Race) and a case where fourth graders were investigating a context where the relationship between partative and quotative division arises, Exploring Soda Machines (Dolk and Fosnot, 2005). The research discussed in Section 4 examines the relationship between the undergraduate's location on the covariation-function landscape and their analyses of the fourth grade work.

During the third iteration of Math 181A (N = 16), the instructional sequence was modified slightly to include additional investigations related to function and covariation and cut back a bit on analysis of children's work. The same three investigations used in the second course iteration were used again, except their order was changed, with the ODE investigation preceding the Cavalieri's investigation. The identical writing assignment involving Frank and Doris' Race and Investigation Soda Machines used in the second iteration was assigned at week seven, this time after completing all three investigations. Their work on this assignment was analyzed and compared to that of the second iteration of Math 181A.

The paper is organized as follows. In Section 2 we explain our perspective on function and covariation and offer a brief review of some of the literature that informed the project. Section 3 outlines our research findings and describes the big ideas on our landscape stemming from Math 181A students collaborative inquiry. In Section 4 these same undergraduates' work in analyzing the progression of ideas in children is discussed. It references some robust misconceptions served as cognitive obstacles to both their problem solving and how they appear to affect their abilities to make sense of children's mathematical development. These issues are at the heart of the paper, for we believe their resolution is essential to, indeed an essential component of, their developing MKT. In the Section 5 we present the landscape (Figure 5.1), which we see as an evolving document, in fact one in which as Fosnot and Dolk hypothesize is better constructed by participants themselves in pre- and in-service courses than handed down by any authority. We will not argue that curriculum needs to be sequenced according to an order of presentation suggested by this landscape—in fact in our courses we have chosen not to do so for other reasons. But we do believe that the landscape indicates growth relevant to pre-service development of MKT and we believe addressing this landscape should be valuable in planning instruction. Finally in the Section 6 we offer some reflections on the challenges in teaching a course that embeds problem solving at a collegiate level with case study of children engaged in investigation.

Section 2. The Function and Covariation Landscape.

For the purposes of this paper, we distinguish between *function* and *covariation* as follows. When discussing function we will consider functions $y = f(x)$ of one variable x with specified domains and ranges, e.g. we take the Dirichlet notion as a concept definition. In fact this definition may not correspond to the concept images of the learner or possible understandings they may hold of a function as an action, process or object, but it serves for the discussion in this paper and was explicitly used in the course. When discussing covariation we will consider a pair of functions $y = f(x)$ and $z = g(x)$, where formally covariation describes a relationship between y and z that holds for all (or specified values of) x. For example, a functional equation $\mathrm{H}(y,z) = 0$ for all x describes covariation, as in the case of a first order differential equation where one might have $y = f(x)$ and $z = f'(x)$. In this way recursive definitions are captured by covariation as the expression of recursion, $f(x+1) = f(x) + g(x)$ (which defines $f(x)$ for all natural numbers x provided $f(0)$ and $g(x)$ are known.) For example if $g(x) = 3$ then a recursive definition of $f(x) = 3x + 1$ for the domain of natural numbers is captured by the condition $f(0) = 1$ and as $f(x+1) = f(x) + 3$ (this example is explicitly studied in Math 181A). As the subjects in this research were upper division mathematics majors it is not unreasonable to expect they might learn to use this formalism as a framework for discussing covariation and function. They were provided readings that made these definitions explicit and the instructors worked with them to help them assimilate the distinctions. In fact it was difficult for them, and the extent to which they could make sense of these constructs and the role this understanding played in their ability to make sense of children's work is our primary interest.

Background on K-16 Concepts of Covariation

Over the past two decades mathematics educators have investigated the role of covariation across K-16 education. Of course, covariational understanding in K-12 takes on forms quite different than our formal characterization above. According to Thompson, "images of covariation are developmental and educators can work on building the covariation concept at any age. In early development one coordinates two quantities' values—think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that in one's image, the two quantities' values persist. In case of continuous covariation, one understands that if either quantity has different values at different times, it changes from one to another by assuming all intermediate values." (Thompson, 1994; Saldanha & Thompson, 1998, p 2.) Although the majority of studies on students' concept of covariation concern calculus and differential equations, some studies described students' early conceptions of covariation in the elementary and middle grades. These later studies provided the motivation to investigate pre-service teachers abilities to recognize children's developing ideas related to covariation and function.

Covariational reasoning for Pre-K to Grade 12. The concept of covariation at the most primitive level can be found in pre-kindergarteners with no counting abilities when the task is set as a part-whole relationship in unknown quantities. Irwin (1996) studied children ages 4-7 and showed that the ability to predict changes to counted quantities increased with age although in her study only 7-year olds were able to use covariance and compensation in the purely numerical context of derived equations. Covariation in part-whole relations can be explained as follows: when the whole consists of two parts and there is a change in only one part, then there is the same amount of change in the whole; Algebraically speaking, if $P_1 \pm P_2 = W$, then $(P_1 \pm x) + P_2 = W \pm x$. According to Irwin, this study provides evidence that in contrast to Piaget's previous claim the concept of covariation or compensation can be introduced at early age before children develop the concept of seriation (Piaget, 1965).

The concept of covariation is also embedded in the multiplicative structure of numbers. For example, when a child is shown seven rows of blocks with three blocks in each row, the child might notice a pattern of three in each increment of row. If the child has no other schemes but basic counting scheme he might just count the blocks by tagging them one by one. If the child notices the pattern of increases by three and has developed repeated addition strategy, he might perform the iterating process and develop a sequence of numbers 3, 6, 9,…, 21 by realizing that 3+3=6, 6+3=9,…, 18+3=21 to find the answer. If the child has developed unitizing scheme, the child understands that 3 blocks together forms a new unit for counting and sees the entire block as seven threes, which is multiplication of seven and three. The child using the iterative process is using a covariation approach to the problem and the child using the unitizing scheme uses correspondence (or function) approach. If we keep adding more rows continuously, we will have an algebraic equation $y = 3x$, with the variable x representing the number of rows and the variable y representing the number of blocks. What does the number 3 represents in the expression? Such a context is used in our research on undergraduates' abilities to make sense of children's thinking (see Section 4 below.)

The study of Blanton and Kaput (2004) examined functional reasoning of children in grades pre-K to 5 with a contextualized problem that can bring forth the relationships between the number of dogs, the number of dogs' eyes, and the number of dogs' eyes and tails combined. The study found the noticeable differences among children depending on grade levels: pre-K children used basic counting without noticing any pattern even after the teacher showed the data by t-chart; grade K children used dots and bars to represent eyes and tails to collect data and noticed a pattern of two and used the words such as "count by 2s" and "every time we add one more dog, we get two eyes"; first grade children used a t-chart on their own and described patterns using "count by 2s" and "count by 3s" . With the teacher, they were able to predict the numbers of eyes for 7 dogs using skip counting and saw the pattern of "double" and "triple"; second grade children used t-chart on their own and described multiplicative relationship using the word "double" and predicted the number of eyes and the number of eyes and tails together for 100 dogs; third grade students used t-charts fluently and represented the data by a graph and described the multiplicative relationship using general words, such as "it doesn't matter how many dogs you have, you can just multiply it by 2"; and the fourth and fifth grade students had similar work to the third graders using only a few data (up to 3 dogs). This study reveals that children at early age, as early as kindergarteners, has the concept of covariation on the counted quantities when the problem is contextualized and the pattern is directly proportional to the step number, meaning that it leads to the relationship, $y = mx$.

Pattern problems which lead to a linear function, $y = mx + b$ with b nonzero are commonly used in the middle school curriculum as a route to transition from arithmetic to algebra and to develop function concept. Studies have found that both upper elementary and middle school students had problems in providing the values for later terms and an expression for the general term (Bourke & Stacey, 1988; Stacey, 1989). Such problems The Math 181A students studied the Case of Ed Taylor where this happens (Smith et all, 2005), and the case provides an opportunity for them to reconsider their ideas about covariation and function.

<u>Covariational reasoning in Calculus and Differential Equations</u>. As noted above the notion of the derivative and indeed the meaning making of differential equations require covariation. The literature on these topics is vast, so we offer only a few comments informing the instructional decisions made in the current research. Thompson (1994), while studying students understanding of differentiation and its relationship to rates of change noted the importance of ratio and proportionality in students ability to make sense of these ideas. For this reason the contexts for covariation in children's work selected in this project are those involving proportional reasoning (or early forms thereof). For the undergraduate part-whole relationship, an inflation rate problem, quoted by Thompson (1994), was modified to fit our needs (see Section 3). Although the problem was introduced in Thompson's article for the purpose of bringing up cognitive obstacles in understanding multiplicative structure of exponential function as a recursive process, the context had a potential in bridging the previous topic of covariational reasoning to the use of part-whole relationships.

The study of functions and their derivatives in the undergraduate years is intimately bound with activities involving graphing. Dekker (1991) describes how iconic interpretations of graphs (where the aspects of graphs are construed as a literal interpretation of the situation) lead to cognitive obstacles in their interpretation and use—many of which we have noted in Math 181A students, for example position with locations on a velocity vs. time graph. Monk (2003) noted that the act of graphing can enable students "to construct new concepts by beginning with important features of the graph" and this strategy has been used in construction of our instructional task involving covariation in locating position when velocity information is provided.

Studies have shown that even high performing undergraduate students hold a narrow view for the concept of function and have difficulty in representing and interpreting real-life dynamic situations that can be represented as functions (Thompson, 1994; Carlson, 1998; Confrey & Smith, 1995). Responding to these concerns are the recent studies focused on students' development of the function concept by developing their covariational reasoning (Smith, 2007; Carlson, 2002, 2003; Confrey, 1994; Confrey & Smith, 1995, Rasmussen, 2000, 2007). Inquiry-based approaches have proved effective in bringing forth (Rasmussen & Blumenfeld, 2007), and these approaches influenced the development of the exploration involving inflation discussed above.

Section 3. Three Investigations relating Covariation and Function.

This section describes three investigations posed in Math 181A whose contexts use covariation to describe a functional relationship. The first is geometric and of historical interest in connection to the development of proportional reasoning, while the second two developed for the second and third iterations of Math 181A are calculus based "continuous" problems. The research uncovered the typical obstacles and resolutions developed by the undergraduates. Analysis of certain transitional moments enabled us to identify emergent big ideas for building our landscape and as formative assessment. From an instructional perspective the purpose of these investigations is to enable the emergence of these big ideas in the undergraduates, to have ample discussion of the ideas and their emergence during class "congresses" in order that they can reflect upon their own mathematical development and to prepare them for investigating similar development among learners in grades 4-8.

First investigation: Functions and Covariation in the a Geometric Context involving Ratio and Proportion: Interpreting Eudoxus' statement, "Circles are to each other as are squares on their diameter."

The dominance of topics related to proportional reasoning in the middle school curricula is well-known, including extensive work with fractions, proportion, and scale, with a goal of developing their interrelations as part of practical problem solving and pre-algebraic reasoning. In order to examine the possible reconstruction of these basic ideas in an advanced collegiate setting, this investigation requires undergraduates to interpret the statement attributed to Eudoxus, "Circles are to each other as are squares on their diameter." The full text of the investigation is presented in the Appendices, but the basic instruction is simple: Explain in modern terms the mathematical result being described.

The first obstacle faced by undergraduates is to decide the statement deals with area. We won't detail the digressions prior to this realization except to mention that there are several typical representations of the geometry based upon a literal interpretation of the sentence. The use of the word "square" usually leads undergraduates to consider area, and often they conclude on the basis of this word that Eudoxus is stating that $A = \pi(d/2)^2$ without considering the special role π plays in this formula. The use of "to each other as" (which we recognize as covariation) does lead to important conversation. Through such conversations a big idea relating to covariation emerges, as one student writes,

> "We started drawing squares around circles because this square represented the area d^2. Then we had a revelation. Eudoxus and his fellow ancient mathematicians did not have the quantity pi. The only way they had to measure a circles area, was in respect to one another. Then we started taking arbitrary circles and calculating their area and comparing the ratio of the areas. We immediately noticed this ratio was the same as the ratio of the squares of their diameters."

Reflected in the change in their approach we find the emergence of a big idea related to covariation, that *the covariation between two functions can be used to characterize key properties.*

The statement gives no indication that Eudoxus understands the role of π in this situation, and during the inquiry the instructor questions the undergraduates about this point. Unlike the student just quoted, others are puzzled by this line of questioning indicating that the big idea of covariation implicit in the statement is at best on their horizon. Their typical verification of Eudoxus statement goes something like this:

$$C_1/C_2 = \pi(d_1/2)^2/\pi(d_2/2)^2 = (d_1)^2/(d_2)^2 = A_1/A_2.$$

Although implicit in this verification is the big idea that *covariation describes the relationship between two functions*, in the end these undergraduates could not distinguish the difference between the two statements "$A = \pi(d/2)^2$" and "There exists a constant k such that $A = kd^2$", nor could they articulate which statement better captured the information in Eudoxus sentence. Nonetheless the interpretation provided by such students is that of covariation and at this point two big ideas linking covariation and function have appeared on the landscape.

Also emerging is an understanding that the covariation is linked to the variation in the area function. A second student discusses in her paper four drawings created during the class inquiry (only at the end of which did the consideration of area emerge) and then in describing a substantive transition in her thinking says,

> "But I still didn't have a precise translation of the problem, as required. I finally realized in the math congress that the correct translation of the statement was 'The scale factor from one diameter to another determines the difference in Area. Specifically, the scale factor squared determines how much the bigger the area of the second circle is from the initial circle.' I was so thrown off from the word square that I thought it meant a real geometric square when in fact it was only talking about a square, as in opposite of square root."

Here we find a second stream of ideas of covariation, namely linking it to variation, as reflected in the choice of word "bigger" (this quotation in the students paper was extracted from a poster presentation during the congress.) The understanding that *covariation can encode the changing value of a function* is emerging for this student (as she reflects upon her notes taken during the class discussion), with a further evidence that the student is thinking about variation provided by her reference to the distinction between a geometric square and the square function.

Finally, the notion that Eudoxus might recognize that there is a constant k for which circular area C = kd^2 less frequently emerges in this investigation. We conjectured this requires ideas about generalizing across classes of functions, for example realizing all functions of the form kd^2 would be related to one another by the property recorded by Eudoxus. This big idea, that *covariation can describe properties of a class of functions*, is important and we decided it belongs on the horizon of our landscape. It should be familiar to upper division mathematics majors, especially if they recognize that a first order ordinary differential equation is an expression of covariation, and our interest in contexts that might lead to its reconstruction led us to develop the next two investigations.

Transitioning to Functions and Covariation in the Continuous Setting I: A Geometric Context involving Cavelieri's Principle

The complexity in the reemergence of the big idea that *covariation can encode the changing value of a function* which had emerged in the Eudoxus problem led us to consider contexts where an interpretation of the rate of change would provide critical information about the function. So a problem about a water tank, shaped as an upside-down pyramid was posed, where the context required finding an expression that relates the rate of water used to rate at which the water level in the pyramid drops (Appendix ZZ). In fact, although somewhat open-ended, this context poses a related rate problem, a familiar routine to students in differential calculus. The derivative of the volume as a function of height is the cross-sectional area (as is checked by the chain rule), which in the integral form is essentially Cavalieri's Principle. Conversation about this point was to provide discussion about covariation, namely as $V'(h) = A(h)$. Further, the calculation of volume as the integral V(h) =∫_0^h A(h)dh when understood as approximated by a Riemann sum, illustrates how the changing value of $V'(h)$ can be understood as determined by the cross-section area, a form of the expression of covariation we sought to develop.

The preponderance of 181A students did not use tools of calculus to solve the problem, and instead made discrete calculations using their knowledge of the volume of a pyramid. In class presentations a typical poster would read,

> The total volume in terms of height is $V(h) = (4/27)h^3$ and then $\Delta V(h) = (4/27)(h^3 - (h - \Delta h)^3)$ is our change in volume from a starting height h and a loss of Δh in height. To lose the same amount of water at 30ft as 60 ft loss with 1 ft, the change in height will need to be greater.

Variations on this poster might include sample values water use for various (seemingly random) height and change in height values, such as one group choose, $(h, \Delta h)$ = (37, 1/5), (24, 2.3), (11, 5). Although these groups could not explicitly formulate a relationship between the water level and the rate of change, they produced an expression of covariation and spoke enthusiastically about their realization that constant water use requires a greater loss of height at lower heights. Their empirical efforts to uncover the relationship show they understand that *covariation can encode the changing value of a function* and *covariation requires keeping track of the both the stage and the amount of change.* This second big idea is crucial to understanding of the recursive definition of functions, proofs by induction, and ultimately Euler's method for approximating solutions to an ordinary differential equation.

There are students who do use the calculus, in particular the chain rule. They usually believe the intent of the context includes the assumption that the water consumption is constant and based upon this determine a relationship between d*h*/d*t* and *h,* sometimes expressed as covariation (as in the Eudoxus problem) but other times expressed explicitly calculated the derivative. Their posters typically read as follows.

> First we calculated $V(h) = (4/27)h^3$ and then $dV/dt = (4/9)\ h^2\ dh/dt$ where dV/dt is the rate of change of the volume with respect to time and dh/dt is the rate of change of the height with respect to time. Suppose the water consumption is constant:
>
> $(4/9)\ h\_1^2\ dh\_1/dt = (4/9)\ h\_2^2\ dh\_2/dt$ which implies
>
> $h\_1^2\ dh\_1/dt = h\_2^2\ dh\_2/dt$
>
> Let $h\_1 = kh\_2$, then
>
> $(kh\_2)^2\ dh\_1/dt = h\_2^2\ dh\_2/dt$ so $k^2\ dh\_1/dt = dh\_2/dt$.

A refinement of this work by other students, if daily water usage is some constant *k*, leads to an explicit formulation (in the notation above),

$$-k = (12/27)\ h^2\ dh/dt, \text{ or equivalently } dh/dt = -27k/12\ h^2.$$

When asked how the rate of change in volume could be computed in terms of the rate of change of height it is rare that students were able to articulate that $dV/dt = 4/9\ h^2\ dh/dt = (2/3\ h)^2\ dh/dt$ at a given *h* value is the cross sectional area at height *h*. This version of Cavalieri's principle was discussed in class, but was rarely taken up in written assignments based upon the problem. But the records of their problem solving, which even when calculus is used is often not direct, reveals two important big ideas in emergent form. The first it that *related rates are an expression of covariation*, and second is that *if one function is known among two whose covariation in known, then the second function can be understood*. Of course, the observation that related rates are expressions of covariation underlies all work in the calculus using related rates, but it is reconstructed in this context. These undergraduates change course and are able to deal with the condition of constant water use after expressing, "the point is that both the amount of water and the height of the water are changing at the same time" indicating this idea is not at all at the forefront of their problem solving. Even among the students who applied the chain rule correctly to the volume formula, many were not able to offer a solution to the investigation as posed and explain how the expression of related rates could be used to determine water usage from an understanding of the rate of drop of height. Exactly why this obstacle occurs likely has a myriad of possible explanations, but because this phenomena appears in other situations we identify as a big idea the notion that if one function is known among two whose covariation in known, then the second function can be understood. We will see this surface again in the context of using an Euler approximation to approximate the solution to a first order ordinary differential equation.

Transitioning to Functions and Covariation in the Continuous Setting II: Difference and Differential Equations

The problem posed here is to determine what happens to the price of a $2,000 computer over two years if the price is subject to inflation (and deflation) given by $I(t) = 10(1 - t)\%$ where t is the time in years. As noted in section 2, this problem was motivated by the investigation posed in Thompson (1994) In relation to other investigations considered in Math 181A, we were interested whether or not students could formulate the ordinary differential equation that describes this context as a measure of their understanding of covariation in a continuous setting, and to compare iterative and continuous solutions a basis for further development of their understanding of covariation. After the discussion of the context to ensure that the class understood that the rate of inflation was 10% at the beginning, but had dropped to 0% after the first-year and that there was deflation -10% at the end of year 2, a variety of approaches and issues arose. For groups that were stumped, at some point during the inquiry the instructors would intervene providing a short review of compound and continuous interest (along the lines of introductory calculus), prompting them to recall the differential equation $D'(t) = rD(t)$ where $D(t)$ is the value of a deposit at time t and r is the interest rate. At times this intervention changed problem-solving behavior and at other times it did not.

A major obstruction for many undergraduates arising in this investigation was that they confounded a per unit rate of change (given as inflation) with an absolute rate of change. Some students even had difficulty understanding inflation as describing a rate of change, where they interpreted the word "inflation" as meaning the computer was over priced (imagine walking into a store and your buddy says, "Gosh, that is an inflated price.") But after class discussion moved them beyond these issues, we found three basic strategies employed by the undergraduates. The first was to iteratively estimate changes over time intervals they felt they could handle (months or tenths of a year), essentially finding Euler approximation although they didn't use that terminology. The second approach was to integrate the inflation rate over a one-year period to find the average rate of inflation over that year, and then complete a two-step estimate (which in effect was a two-step Euler approximation). None of the undergraduates using these two approaches recorded or otherwise indicated they were solving a differential equation. In many cases this approach persisted in spite of the instructor intervention recalling continuous interest and the differential equation upon which it was based. In the third approach, students recorded the ordinary differential equation suggested by the context and solved it by separation of variables to conclude the price returned to its origin after two years. A number of the undergraduates using the first approach initially did not take into account the time interval (part of a year) in which to compute the inflation, but this issue was usually resolved by intervention.

Illustrating the first approach is a congress poster that reads,

> We calculated the price each three months, $(2000)(1+ .075/4) = $2037.50, $(2037.50)(1+.05/4) = $202.97,…, with a final value $1948.93 after two years.

Their explanations reveal an understanding that *per unit change is an expression of covariation*, that *covariation can encode the changing value of a function* and that *covariation can iteratively determine the value of a function.* This latter idea provides a basis for making sense of Euler approximation. Although a variety of time intervals were considered, interestingly some students assumed that it was not necessary to carry out the calculation because the negative inflation rates would "cancel out" the positive rates, and therefore asserted that the final price would return to $2,000 (the correct continuous value, but not the result of the discrete calculation they presented.) Prompting students to check their hypotheses let them to realize that *covariation requires keeping track of both the stage and the amount of change*, essential to making sense of the iterative feature of Euler approximation.

The following poster exemplifies the second approach,

> To find the inflation of the first year we must integrate our rate of change of inflation equation for the first year $\int_0^1 I(t)\, dt = 5\%$ and for the second year $\int_1^2 I(t)\, dt = -5\%$. So the price after two years is (2,000)(1.05)(.95) = 1,995.

Although only leading to a two-step discrete approximation, this approach resulted from extended conversation and was viewed by the groups as a substantial improvement over multi-step Euler approximations. An emergent big idea reconstructed in this work is that *the derivative is a continuous expression of a rate of change*, and this idea provided the basis for their decision to integrate the function I(*t*) to find its average value (which they did symbolically, not graphically).

The third approach sets up the expected differential equation.

> We set up the ODE $P'(t) = .10(1 - t)P(t)$ and solved by separation of variables to find that $P(t) = 2000e^{(.10t-.05t^2)}$. From this we found that there was no change in price after two years.

These undergraduates were typically stuck for a while, often starting with discrete approximations until the big idea that *differential equations are expressions of covariation* emerged form their discussion of context. In the congress the participants were asked to compare the first and third approaches. Typically this was difficult, but some were able to articulate the big idea we hoped would be on the horizon, that *difference equations are discrete versions of differential equations.*

Section 4. Analyzing Children's Work

Functions and Covariation in the Discrete Setting: Figurate Numbers

Many middle curricula include problems that expect students to describe and extend geometric patterns, where the total number of objects might constitute an arithmetic progression or in a more complicated case be represented by a polynomial function. Border problems (the number of square tiles required for a square or rectangular border) or the calculation of triangular numbers resulting in $T(n) = n(n+1)/2$ are popular examples. In each case there appears to be an evolution of ideas, likely stimulated by the context of using a sequence of shapes, where the "step by step transitions" are described first (a covariational description of the function) and an "overall function expression" is obtained later, if at all[2]. In Math 181A figurate number contexts again provide an opportunity for students to construct the idea that *covariation can encode the changing value of a function* and are used in two ways, first as further inquiry, and second as an opportunity to investigate children's work, which in this case was based upon the study Examining Nonlinear Growth Patterns—The Case of Ed Taylor (Smith et. al. 2005). These problems also provide a context for undergraduates to revisit proof by induction (part of a transition to upper division mathematics course at UCSB) and are used by many secondary and undergraduate curricula for this purpose. Indeed, making sense of proof by induction requires the big idea we revisit below, that *covariation can iteratively determine the value of a function*. At the conclusion of work with Ed Taylor's case and similar figurate number homework problems, the 181A students in the first course were expected to write about commonalities between them and their work translating the Eudoxus statement (these papers are a key source material.) It is through these writings that developing MKT can be examined.

In the case, Ed Taylor's students are analyze a border problem, and then analyzing the figurate "S-pattern". Math 181A students began analyzing the same figures and discussion from a congress on S-pattern illustrates how their development of a description of covariation precedes their description of the function. The opening presentation began with the following explanation (on their poster) of how the S-pattern grows,

> Add the current step number to the right hand column, then a row of the next step number to the top.
> Number of boxes added/growth: $n + (n+1)$ where n is the current step number.

The explanation lacks recursive notation for the total number of squares, but the big idea that *covariation can iteratively determine the value of a function* is evident in the discussion. Although the presenters were able to locate both incremental components, n and $n + 1$, when provided the notation $S(n)$ = total number of squares in figure at step n, with prodding they are able to formulate a recursive rule $S(n) = S(n - 1) + n + (n+1)$,

[2] The terminology "step by step transitions" and "overall function expressions" used here may not be standard, but has been used by the first author for years in professional pre- and in-service work.

explaining to the class that "S(n – 1) is the total number of boxes at the previous step and n + (n+1) is the growth from the previous step to the next step." A student from the class suggested the equation should read S(n) = S(n – 1) + (n – 1) + (n – 1 + 1), noting that their expression does not yield the correct values when n = 2. A third student agreed, pointing out that the latter formula can also be understood as S(n) = S(n – 1) + n + (n – 1). This is then compared to S(n + 1) = S(n) + n + (n + 1) where it is noted that this latter expression matches the language on the poster, of "adding the current step number (to the right hand column and a row of) the next step number (to the top)", which gives the next figure, at which point it becomes evident to the presenters (through smiles and relief) that recursive expression makes finally makes sense. Here in the congress the big idea that *covariation requires keeping track of the both the stage and the amount of change* emerged during the presentation.

The second group in the congress described the general figure at Step n as having (n – 1) columns and (n + 1) rows, so that the total number of squares is (n – 1) x (n + 1) + 2 (which via the geometry in the S-pattern can be seen as equivalent to n^2 + 1, although not yet raised). A third group represented the figure at Step n as an (n – 1) x (n – 1) square with two rows of length n attached, so total area is (n – 1) x (n – 1) + n +n = n^2 + 1, and when the students uncover the same geometric reconfiguration of the S-pattern as an n x n square with one more, they again show excitement and surprise that this can be done. At this point in the congress the question was posed about how the formulae S(n + 1) = S(n) + n + (n + 1) and S(n) = n^2 + 1 are related. It was not obvious to the class. In fact the big idea linking covariation and function, that *covariation can provide the recursive definition of a function*, is not yet on the horizon. We were interested if some linkage could be forged, for example something like $((n + 1)^2 + 1) – (n^2 + 1) = 2n + 1$, some form of which is essential to an inductive proof. Although some of the undergraduates came close, none actually made this connection.

Following the class discussion of their collaborative work of the S-pattern, the class examined and discussed the case presented in the Smith et al materials. They subsequently wrote about both the Eudoxus problem and the role of covariation and function in the work of the students in Ed Taylor's class. They were explicitly asked to "identify some of the big ideas, strategies, and representations that emerge in the student work" and were expected to provide evidence based upon the language used by students. The examination of their work shows their ability to locate the relationship between the covariational expressions and the function in the student's discussion takes on emergent forms. But for the most part their writing illustrates a compartmentalized understanding and don't link the two, or often focused on the use of variables rather than the strategies and ideas related covariation and function:

> There were three big ideas I was able to find in this problem Mr. Taylor's class did with the S-pattern. With the visual aid the student would use grouping to help them count the squares and the students began to have an understanding of arrays for the S-pattern. Lastly, relating the sequence of blocks sequentially verses understanding the sequence as a whole.

> The big idea that was used with these strategies was the idea that an expression can find the total number of something even if the value changes as long as the values all have something in common.

> They were tasked with effectively describing how to generalize steps in the pattern and how to quantify the number of tiles that make up the pattern in any step. One student expresses the awareness of this big idea, referring the Ed Taylor case,
> This activity was used to introduce the concept of algebraic abstraction. This is where quantities and the ways quantities can change are represented as variables and equations. This could definitely be thought of as a big idea of math because it represents a change in the way the math must be thought about. That's just what a big idea is, a shift in perspective or logic.

> Charles understands use of arrays, and use of arrays is a big idea.

However, some 181A students noted that Ed Taylor's student Michael locked into the covariational approach and were able to explain the relationship between the spatial representation and the big ideas,

> Michael did not discover a particular equation for solving the total number of tiles in the nth step, because he said that he needed to know that previous steps' pattern to get the present steps result. He said that each pattern had an additional column and row sticking out. This is a good way for seeing how the new patterns will look, but it is not helpful in showing the general way of doing the problem.

The writings on Ed Taylor were evaluated according to the following rubric:

> Level 1: Does not discuss Michael's covariation approach and focuses only on correspondence.
> Level 2: Discusses both correspondence and covariation, but does **not** validate covariation as a valid developmental approach. For example, see covariation as a less intelligent approach or just different kid of approach
> Level 3: Discusses both correspondence and covariation and validates covariation as an appropriate developmental approach in some way.
> Level 4: Further, connects covariation and correspondence mathematically, for instance the step by step increment provides a basis for an inductive proof.
> The papers were also coded according to two further criteria:
> M: Discusses use of model as a justification.
> SO: Discussion focused mostly on symbolic aspects (for example parentheses) rather than the model.

Twenty of the 29 class students wrote on the work of Ed Taylor (the others chose to write about a different case.) All but three discussed how the children used the model for justification. Eight of the twenty (40%) were scored as Level 1, meaning they failed to discuss Michael's covariational strategy, which in fact, was the first strategy chosen by

Ed Taylor for class discussion in the case. Nine of twenty (45%) were scored at Level 2 meaning they were able to distinguish Michael's approach from that of the other children in the class who developed some form of $n^2 + 1$ as a description of the total number of tiles. But they did not offer any indication that these approaches were connected, either developmentally or mathematically and at times devaluing his thinking with comment such as his work "was not helpful". Three of the twenty students (at Level 3) offered some form of validation that Michael's work belonged in the landscape and none reached Level 4.

In a subsequent homework, the students were provided the article by Eric Smith (2007) on covariation and variation to provide for them relevant vocabulary and definitions, and they were asked to define covariation and locate an example of it in the Ed Taylor case. Twenty-six students completed the homework, and all but five could produce a reasonable description of covariation (either quoting the article or from another source) but only fifteen correctly identified Michael's work as an example of covariation. Interestingly, many of the students who discussed Michael's strategy in their paper, were not subsequently able to see it was an example of covariation.

Functions and Covariation in the context of Partative and Quotative Division.

The second iteration of our hypothetical learning trajectory included the two calculus-based investigations described in Section 3, and a new measurement of undergraduates' ability to interpret the role of covariation and function in children's work. This revised assignment paper included a problem they would solve individually (Frank and Doris' Race) and a transcript of a Congress discussion involving two children (AnaMaria and Kevin) presenting their thinking about the relationship between two division contexts, one partative and the other quotative. The hypotheses guiding these changes were two-fold. First we believed that calculus-based contexts where the instructors would explicitly link covariation and function could be simultaneously more problematic and meaningful to the undergraduates. Second, because AnaMaria and Kevin's presentation was specifically devoted to explaining the relationship between partative and quotative models of division where their covariational and functional approaches emerged, we believed it would encourage the undergraduates to examine the connections more effectively than the compartmentalized class discussion in the Case of Ed Taylor. As noted, during the third course iteration, this assignment was postponed until later in the term allowing for, hopefully, greater success.

The undergraduates were specifically asked to discuss covariation in these two contexts. In the Frank and Doris problem they are comparing two velocity functions and the relationship between these two velocity functions is covariation. One issue is if they cofound the covariational relationship involving rates of change with position (total change). More specifically we hoped they might comment on velocity vs. time, to distance vs. time as an example of covariation (and the Riemann sums involved as Euler approximations, like the inductive step). In AnaMaria and Kevin's work, the children are working on the two problems: (1) If a coke machine has 156 cokes, how many 6-packs would it take to fill the machine?, and (2) If a soda machine has 6 flavors and has 156 cans when full, how many of each flavor are in each column of the machine? (Dolk and Fosnot 2004?) During the Math 181A discussion, AnaMaria and Kevin's work was discussed, and it was noted that they started with the second problem where they attempted to check if there could be 18 and then 23 cans of each flavor using a repeated addition strategy, and subsequently note that each time they add a pack the sum increases by 6. IN the writing prompt, the undergraduates are asked to link the expression f(x) = x+x+x+x+x+x to AnaMaria and Kevin's initial work to the recursive expression f(x+1) = f(x) +6, and to discuss the evolution of their thinking and the relation between partative and quotative division to covariation and function.

Work on Frank and Doris. The problem Frank and Doris' Race is a typical introductory calculus problem designed to raise the possible confounding of rates of change and total change in a motion context. Although Doris' initial velocity is larger than Frank's, they have the same velocity at ten seconds after which Frank's velocity exceeds Doris' and he will eventually catch up. The question, based on numerical data, is who will win the ¼ mile race and when will Frank pass Doris. The numbers were chosen so that most any realistic estimation, say a trapezoidal approximation, will show Doris wins the ¼ mile race.

The range of responses among 181A students to Frank and Doris were similar to the range one expects among first year calculus students as exemplified below. Some of the undergraduates confounded the relationship between speed and distance travelled, "Frank passes Doris at 10 seconds and wins." This undergraduate made this claim although he also mentions area under curve as representing distance. Other undergraduates recognize that the area under the curve does represent total distance travelled (and typically used a trapezoidal approximation which they typically described as using average speeds over each five second interval), but do not appear to think about distance as a function of time so they miss the covariation. Even though they found (comparing areas) that Frank's distance equaled Doris at some point well beyond the ¼ mile mark, they had to calculate the time each reached the ¼ mile mark to determine who won the race. There were undergraduates who felt that since they did not have an explicit function for velocity vs. time they either could not solve the problem, or in order to do so they had to use some curve fitting technique (in a few cases they relied on a computer program for this). Again in this case, as the students integrated a velocity function symbolically (or with a machine) they did not communicate an understanding that covariation in this case could be thought of as providing a step by step accumulation of area under the curve (distance). There were undergraduates who did identify the covariation as a relationship between the two functions of velocity and distance travelled with the velocity being used to iteratively determine distance traveled.

Work with AnaMaria and Kevin's thinking. Each class viewed the introduction of the context by the teacher (Tanya) and then viewed children at work, among them the pair AnaMaria and Kevin. The class discussed the difference between the partative and quotative division and how the context of the soda machine led to the array model and the two questions (How many six packs are in a machine with 156 cokes? and How many of each flavor it a machine with 156 sodas has six flavors?) The fact that the machine had six columns was also part of the children's and the Math 181A class discussion. In the writing assignment the 181A students had a copy of the transcript of AnaMaria and Kevin's presentation in the Congress and were supposed to discuss the role of function and covariation in their work. In this transcript Kevin begins the discussion as follows.

> Well what me and AnaMaria did was, um, we used the first on for, we used the second one to do both of the problems. We begin with the second questions, and so we drew a soda machine and we, um, named all the different sodas—kinds of sodas there were. So we … when we … when they was … 6 times 12 there was … 72, we knew it wasn't that much so it, we knew 6 time 18 was 108, it wasn't … so we went to 23 six packs, 138, 144, then 150 and then we just needed six more to add—to add six more. That's how we got our answer.

Most of the undergraduates understood that $f(x) = 6x$ represented in this context a function, while $f(x+1) = f(x) + 6$ represented covariation (they had a hand out that spelled this out in another context). But often they confounded which corresponded to the problem or strategy the child was using claiming that AnaMaria and Kevin only added on sixes using covariation, in contrast to Kevin's opening statement. However a good number were able to identify the development in AnaMaria and Kevin's thinking and how their insight of the covariation in the multiply-by-six function $f(x) = 6x$ provided them an understanding of how partative and quotative division were related.

Results from Analysis.

Patterns emerged in the analysis of the work on this paper as is noted below. In analyzing the undergraduates work on Frank and Doris' Race the following rubric was used.

> Level 1: unable to solve the question or use position function formula from physics or mathematics class.
> Level 2: Solve the problem, but unable to see function is developed from covariation concept in their work, i.e., step by step accumulation of areas under the curve.
> Level 3: Solve the problem and see function is developed from covariation concept in their work, i.e., step by step accumulation of areas under the curve.
> The papers were also coded according to three further criteria:
> O: obstacle in interpreting speed function, i.e., 10 seconds Frank overtakes Doris.
> M: distance as area under the curve.
> A: use average speed to find the area under the curve for 5-second interval period.

In analyzing the undergraduates' work on AnaMaria the following rubric was used.

> Level 1: mention covariation and function at abstract level in words or forms, such as covariation is …function is …, or $f(x+1)=f(x)+6$ for covariation and $f(x)=6x$ for function), but not within the context.

Level 2: notice covariation as f(x+1)=f(x)+6 and function as f(x)=6x within the context, but the connection is not clear.
Level 3: notice covariation and function concept in AnaMaria's second problem and provide the relationship between them.

The papers were also coded according to five further criteria:
U: mention unitizing 6 packs or 6 flavors to explain covariation or correspondence view.
D: discuss children's understanding of division through multiplication.
P: discuss the relationship between quotative and partative division.
M: discuss role of area model in the context.
R: discussion on how AnaMaria and Kevin's work and their work on Frank and Doris are related in terms of covariation and function.

Table 4.1 summarizes the results of the evaluation of papers during Winter 2009 (N = 22) and Fall 2009 (N = 15). The first number indicates the rubric score for Frank and Doris' Race and the second AnaMaria and Kevin's work.

Table 4.1

	1,1	2,1	3,1	1,2	2,2	3,2	1,3	2,3	3,3
W 09	5	4	1	0	0	0	0	7	5
F 09	4	5	0	2	2	0	0	1	1

Two aspects of this analysis are striking. The first is that in both groups, the ability of an undergraduate to differentiate the roles of covariation and function in their own work correlates highly with their ability to find it in the work of children (this is indicated by the low number of scores in the center of the table.) The second is that the W09 undergraduates were substantially more successful than the F09 students on this task, in spite of the fact that the instructional materials were redesigned for F 09 to test the hypothesis that more work with contexts involving covariation in the continuous setting would lead to improvement. (Recall that the F 09 undergraduates studied the Cavalieri's Principle and Differential Equations Problems prior to writing the paper, while in W 09 the situation was the reverse. As it played out, this meant that F 09 undergraduates spent much less time studying the strategies and models that underlie basic number sense, such as the area and ratio table models. In fact in their final exam, many F 09 undergraduates confounded ratio tables with general in-out chart for a function, something that would not have happened if more time had been spent on these issues in the course.) So although the classes had different personalities, we do suspect that the emphasis on continuous mathematics as opposed to the discrete, as well as resulting loss of emphasis on the developmental models of number sense, prompted the F 09 undergraduates to either create a continuous velocity function to reason directly with area and thereby miss the iterative nature of calculating the distance function. So then, when asked to link the approaches to the work of AnaMaria and Kevin, they missed the transition in their work as well. In fact, in much of their writing it appears as if they missed the fact that distance traveled is a function—instead it is a numerical value to be found only in the context of the problem.

The additional coding reveals that most of the undergraduates understood distance traveled as area under the curve, and that few were trapped by the cognitive obstacle confounding speed with position. There was a striking difference, however, in their observations about AnaMaria and Kevin's work. Most of the students in W 09 commented in details about the role of the array/area model for multiplication and the role of unitizing six packs to explain the covariational approach, while essentially none of the F 09 students did. Although both groups discussed the difference between partative and quotative division, it seems that the W 09 students' familiarity with covariation in discrete contexts and with the models that underlie multiplicative number sense gave them the upper hand in discerning AnaMaria and Kevin's development.

Section 5. The Covariation-Function Landscape.

In section 3 the following big ideas were identified as playing important roles in changing problem solving behavior:

- *the covariation between two functions can be used to characterize key properties*
- *covariation describes the relationship between two functions*
- *covariation can encode the changing value of a function*
- *covariation can describe properties of a class of functions*
- *covariation requires keeping track of both the stage and the amount of change*
- *related rates are an expression of covariation*
- *if one function is known among two whose covariation in known, then the second function can be understood*
- *per unit change is an expression of covariation*
- *covariation can iteratively determine the value of a function*
- *covariation can provide the recursive definition of a function*
- *the derivative is a continuous expression of a rate of change*
- *differential equations are expressions of covariation*
- *difference equations are discrete versions of differential equations*

Based upon our analysis of papers described in Section 4 we add three more.

- *components of expressions describing functions given by figurate numbers can be represented geometrically.*
- *variation is necessary when working functions. (Functions are not simply expressions or a static object.)*
- *a loci of points on the Cartesian plane can represent covariation*
- *linearity (proportion) can be understood as constant covariation.*

Big ideas are only one type of landmark on our landscape. Strategies and Models are important too. The landscape of covariation and function is complex, blending discrete and continuous contexts, with a myriad of strategies related to induction and recursion, the chain rule (related rates and integration), or iterative methods that blend these such as Euler's method. The most familiar model is that of a graph of a function, but so are models of measurement; linear—such as the difference model on the real line, area—as a model of multiplication that can represent Riemann sums, as well as their higher dimensional analogues. All of these have been investigated by many authors and will continue to be as subject of ongoing research.

In this article we have outlined our work and evidence that indeed they are big ideas, and in fact are characterized by learner's shifts in learner's reasoning while engaged in problem solving. So in Figure 5.1 below we include our landscape where we place these big ideas along side the strategies and models where we found them. Like Fosnot and Dolk we view these landscapes as evolving documents, the placement of landmarks is not sacred (indeed we disagree somewhat among ourselves) and so we do not offer details as to why certain ideas are located where there are except that it indicates our best sense at this time. The role of the landscape is to draw attention to the role of big ideas in the development of the learner, to inform instructional decisions, and to help explain certain phenomena. Most important, as the metaphor *landscape* indicates, there are myriad of paths undergraduates will take as they reconstruct their ideas about covariation and the challenge of instruction is to design contexts which provide opportunities for growth in these various directions. We believe that the investigations that were researched here offer some evidence that this is possible.

Section 6. Concluding Remarks.

As noted by Thompson (1991), "The tension between thinking of function as covariation and of a function as correspondence is natural. They are both part of our intellectual heritage, so they show up in our collective thinking." The goal of this research was not to offer evidence as to the relative value of pedagogical approaches, but rather to examine the landscape of development of undergraduates (as pre-service secondary teachers) as they begin to grapple with this "tension" and develop concept images of what we mean by covariation and of a function as correspondence. We believe, in short, an appropriate objective for such a pre-service course is to engage them in this conversation, so that as they grow professionally, they will be attuned to the roles of these notions in the development of their students. The research outlined in Section 4 provides evidence that indeed there is a linkage between undergraduates mathematical knowledge in this arena and their developing MKT. Given the continued debate about the relevance of a mathematics degree to K-12 teaching, especially in middle grades[3], we hope that the possibility of linking problem solving to case study analysis will be utilized and researched more fully at the undergraduate pre-service level.

The landscape presented in Section 5 is based upon analysis of pre-service teachers engaged in inquiry on problems related to covariation and function. These students had completed the Calculus and an Introduction to Proof course where they have learned about proof by induction. For most, the formal notion of covariation was new, although they had a variety of experiences with it prior to Math 181A. But their understanding of the relationship between covariation and function was fragile and in the course most of

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[3] For example Cavanagh (2009) reported research "does not show a link between teachers who majored in math and higher student achievement, especially before high school."

the big ideas located on the landscape required reconstruction. The research provides evidence that the hypothesis that introducing upper division undergraduate math majors to ideas about covariation and function would benefit from use of calculus-based examples (because of their "familiarity") does not ring true. More of a middle ground approach may be needed, with more time with discrete examples and more time may be needed for developing the models that underlie basic number sense. We believe such an approach will foster greater mathematical development among these undergraduates as well as greater MKT development for future work involving covariation and function.

Fig. 5.1 Covariation and Function Landscape

Covariation

Functions

I. difference equations are discrete versions of differential equations

M/S: Uses slope fields to construct graphical representations of solutions to ODEs

M: covariation can be expressed in additive or multiplicative forms depending upon context (example y = 3x+1 vs y = 2^x.)

S. uses covariation to represent and estimate y given a graph of y'

I: covariation can iteratively determine the value of a function (such as induction and recursion in discrete contexts or Euler's method in continuous contexts.)

M: area under curve of a rate of change represents total change

I: covariation describes the relationship between two functions

I: related rates are an expression of covariation

M. graph of price vs time and inflation: relationship between part whole relations and rates as covariation

M: Uses T-chart while describing covariation

M/S: Understands and uses slope field as an expression of covariation.

I. the covariation between two functions can be used to characterize key properties.

M: covariational expressions f(n+1) = f(n) + … and f(n) = f(n-1) + … can define equivalent functions

I: covariation can provide the recursive definition of a function (seeing how variables in a pattern

I: linearity (proportion) can be understood as constant covariation.

M: equivalence of expressions describing functions given by figurate numbers can be represented geometrically.

M: uses f(n+1) = f(n) + … notation to denote covariation.

I: covariation requires keeping track of the both the stage and the amount of change

S: keeps track of step numbers to ensure f(n+1) = f(n) + … equations are properly set up and plugs in numbers.

S: generalizes from sample instances of covariation, and records change as an expression (e.g. n + (n+1)).

S: uses subsets of figurate numbers to study covariation (for example finding earlier figure

M: represents general covariational change pictorically (figurate diagrams) or graphically (e.g. showing Δy and Δx increases)

I: variation is necessary when working functions—functions are not simply expressions or a static object

S: uses covariation to find new function values.

I. a loci of points on the Cartesian plane can represent covariation

M,S: draws successive changes to represent covariation.

M: Figurate diagrams are aligned next to each other to indicate relationships (e.g. squares and circles and how they change)

M: Use ratio table to model direct variation

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Appendices

Math 181A Second Writing Prompt, Spring 2008
Paper due Thursday, May 15

In this paper you are to address *the relationship between the development of language and the development of mathematical ideas.* Before you start, we want to make is clear we are looking for content-specific connections. We are **not** looking for a general essay whose premise is "if you know the language you can understand the math better". This latter statement is very true, but you must look deeper than this. We want you to discuss the relationship between the language that inventors of mathematics use (this includes learners) and the development of mathematical ideas by writing about the problems considered in this course.

Part I. In the first part of your essay you need to discuss two of the four problems from the Mathematical Language investigation started on April 24. We completed the work on the three statements of Eudoxus carefully in class, and you must work on at least one other problem of your choice from the three on the sheet. (You can work on two other problems from the sheet if you prefer.) We suggest you work on the additional questions with a classmate, and of course you can talk with Profs. Jacob and Lager.

Your task is to first discuss the mathematical content in these problems in a precise way, and then second to discuss how you interpret the historical language used in the problem statement. Note that the use of language gives you clues into the historical development of the subject. At the same time, the developmental level of the mathematics at that point in history influences the language used. So the mathematical development and language are linked historically. Discuss the implications of all this when describing the mathematics in the problems you selected.

Part II. In the second part of your essay you extend the themes you have developed in Part 1 to one of the two classroom cases studied these past few weeks. You may choose either the submarine sandwich investigation or the Ed Taylor case. If you choose the submarine sandwich investigation we will check out a copy of the CD to you that you can load on your computer. In either case, you first need to identify some of the big ideas, strategies, and representations that emerge in the student work. Look over your previous paper to make sure you are on target with what we are looking for and reread sections of Fosnot & Dolk if you are uncertain. We use these terms in a specific way to focus on mathematical development, not general characteristics of a math class (for example, showing students that there are "lots of different strategies to solve a problem" is not a big idea). Second, extend your ideas from Part 1 and discuss how the language of the learners is related to the mathematical ideas they are grappling with. This is both cause and effect: students' use of language reflects where they are on the landscape of big ideas and strategies, and conversely, the language students use and how they interpret other peoples language can influence how they grow mathematically. Be as specific as possible.

General Comment: A few well-chosen examples, carefully laid out, will take you a long way and can earn a top score. You do not need to address everything, but what you do address should be clear, both mathematically and in writing. We also expect you to adhere to the five-page limit (not including diagrams). Please go back and re-write your first drafts. Many of the first papers needed this, and could have been shortened and clarified a good deal in the process.

Math 181A Paper 1: Due January 26, 2009

In this paper you are to write about the relationship between covariation and functions, and two instances where this relationship can be represented with the area model. You are to solve the problem, Frank and Doris' race, and you are to compare your work on this problem with the class' discussion of the work of AnaMaria and Kevin on the Soda Machine problem. This at first may seem like quite a leap: your work on Jim and Tom's race implicitly uses the fundamental theorem of calculus, while AnaMaria and Kevin are grappling with the distinction between partative and quotative division. But the cognitive issues are related, and you need to make sense of and describe this relationship.

Your paper should include:

1. A solution to the problem about Frank and Doris' race. Identify the role of covariation and function in this problem you should explain the meaning of "area" in the graph provided with this problem, as well as

2. A discussion the transcript of the class discussion of work of AnaMaria and Kevin on the soda machine investigation and an explanation of the mathematical issue the students are grappling with. Identify the role of "area" in the representations of the problem context. While reading the transcript, keep in mind that Tanya is the teacher. We can arrange for you to view this discussion if you like.

3. An explanation of how the cognitive issues involving covariation and function are related to the issues confronting Tanya's class. To help you, think about the function f(x) = 6 x. Note that f(n+1) = f(n) + 6, and f(x) = x+x+x+x+x+x are two ways to understand this function. The area model and Cavalieri's Principle should help you too!

This paper is not to exceed five pages and should be typed in12 pt font with one-inch margins. What you write beyond five pages will not be read, so you must revise and write concisely. We expect you to include diagrams and figures as necessary and cite them by number in your writing, but these should prepared on separate sheets of paper and placed at the end of your paper. The figures do not count towards the five-page limit.

Note: By Thursday January 28 you need to have read Chapters 4 and 5 from Fosnot and Dolk. We realize that you will be busy preparing your first paper, but please save time to make sure you complete the reading by the 28th. Chapter 4 is from the green book. It is similar to the chapter 4 from the Fosnot and Dolk multiplication book, but focuses on par-whole relations instead of the relationship between multiplication and division.

Frank and Doris' Race

Frank and Doris are good friends who both love drag racing (both are engineers). They each bought cars and are busy preparing them for the July 4 Shelbyville race. Both cars are similar, but Frank has installed a small nitrous tank to give him an extra boost towards the end of the race. But it also slows him down earlier on. Below is a speed vs time chart for both cars obtained from their theoretical models:

Time (seconds)	Doris' Speed (ft/sec)	Frank's Speed (ft/sec)
0	0	0
5	50	20
10	85	85
15	120	140
20	130	150
25	140	155
30	150	160

If these model's predictions are accurate, who will win the quarter mile race? When will Frank pass Doris?

MATH 181A: **Mathematical Language Project**

Mathematicians usually distinguish between ideas and the notation used to represent them. For example, we are careful to distinguish between the idea of numbers and the numerals that represent them. Mathematical notation and language has a rich history, and its limitations have had a profound effect upon the subject's development. In this project we want you to reflect upon the language and notation of mathematics.

Given below are some results stated using mathematical language that is close to that used by those who discovered the results (of course these mathematicians didn't use English!) Your task is to make sense of what they mean and to translate these results into modern notation. (You can check to make sure your translation is correct by proving the result!) Think about the language used. Did it limit the ability of the mathematicians to conceptualize their ideas? How about their ability to communicate ideas? Why or why not?

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1. Translate the following results of Eudoxus into modern terminology:

(i) "Circles are to each other as the squares on their diameters."

(ii) "Spheres are to each other in the triplicate ratio of their diameters."

(iii) "A pyramid is the third part of a prism having the same base and the same height."

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2. (Thymaridas, 4th Century B.C.) "If the sum of quantities be given, and also the sum of every pair which contains a particular one of them, then this particular quantity is equal to the difference between the sums of these pairs and the first given sum divided by number of quantities less two." (This is stated in so-called "rhetorical algebra".)