

# Matrix polynomials and structured linearizations.

Advisor: Maria Isabel Bueno Cachadina

Let

$$P(\lambda) = A_k \lambda^k + A_{k-1} \lambda^{k-1} + \cdots + A_0 \quad (1)$$

be a matrix polynomial of degree  $k \geq 2$ , where the coefficients  $A_i$  are  $n \times n$  matrices with entries in a field  $\mathbb{F}$ .

A matrix pencil  $L(\lambda) = \lambda L_1 - L_0$ , with  $L_1, L_0 \in M_{kn}(\mathbb{F})$ , is a *linearization* of  $P(\lambda)$  if there exist two unimodular matrix polynomials (i.e. matrix polynomials with constant nonzero determinant),  $U(\lambda)$  and  $V(\lambda)$ , such that

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

Here  $I_{(k-1)n}$  denotes the  $(k-1)n \times (k-1)n$  identity matrix.

Linearizations of a matrix polynomial  $P(\lambda)$  share the finite eigenvalues of  $P(\lambda)$ , among other important properties. Beside other applications, linearizations of matrix polynomials are used in the study of the polynomial eigenvalue problem  $P(\lambda)x = 0$ . In the classical approach to this problem the original matrix polynomial  $P(\lambda)$  is replaced by a matrix pencil  $L_P(\lambda)$  of larger size with the same eigenvalues as  $P(\lambda)$ . Then, the standard methods for linear eigenvalue problems are applied.

The *reversal* of the matrix polynomial  $P(\lambda)$  in (1) is the matrix polynomial obtained by reversing the order of the coefficient matrices, that is,

$$\text{rev}(P(\lambda)) := \sum_{i=0}^k \lambda^i A_{k-i}.$$

A linearization  $L(\lambda)$  of a matrix polynomial  $P(\lambda)$  is called a *strong linearization* of  $P(\lambda)$  if  $\text{rev}(L(\lambda))$  is also a linearization of  $\text{rev}(P(\lambda))$ . Strong

linearizations of  $P(\lambda)$  have the same finite and infinite elementary divisors as  $P(\lambda)$ , therefore they preserve not only the finite but the infinite eigenvalues. Moreover, if  $P(\lambda)$  is regular (i.e.  $\det(P(\lambda)) \neq 0$ ), any linearization with the same infinite elementary divisors as  $P(\lambda)$  is a strong linearization.

In applications, many matrix polynomials have some structure: either all the matrix coefficients are symmetric, or Hermitian, or skew-symmetric, or the coefficients alternate between symmetric and skew-symmetric, etc. For each structured matrix polynomial  $P(\lambda)$ , many different linearizations can be constructed but, in practice, those sharing the structure of  $P(\lambda)$  are the most convenient from the theoretical and computational point of view, since the structure of  $P(\lambda)$  often implies some symmetries in its spectrum, which are meaningful in physical applications and that can be destroyed by numerical methods when the structure is ignored. For example, if a matrix polynomial is real symmetric or Hermitian, its spectrum is symmetric with respect to the real axis, and the sets of left and right eigenvectors coincide. Thus, it is important to construct linearizations that reflect the structure of the original problem. In the literature different kinds of structured linearizations have been considered: palindromic, symmetric, skew-symmetric, alternating, etc. Among the structured linearizations, those that are strong and can be easily constructed from the coefficients of the matrix polynomial  $P(\lambda)$  are of particular interest.

In this project, we will focus on some open questions related with symmetric linearizations of symmetric matrix polynomials.